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Cycles with Exactly Two Amplifications in GCD-Augmented Fibonacci Recurrences

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Abstract

For the recurrence $C(n) = C(n-1) + C(n-2) + \gcd\big(C(n-1), C(n-2)\big)$, the gcd dynamics is governed by a cofactor map R on coprime pairs, whose cycles are classified by the number of amplifications, the steps per period at which the gcd grows. The author's earlier work showed that exactly three cycles have one amplification and left two amplifications open, as the analogue of 2-cycles in the $3x + 1$ problem, which do not exist. We show that cycles with exactly two amplifications exist. In $\mathbb{Z}[\varphi]$, a cycle multiplier equal to the Lucas number L_q makes the determinant of the cycle equation a unit, so every factorization of L_q yields an admissible candidate and only content conditions decide. We give twenty-five explicit cycles at ten periods, the smallest of period 188, each determined by three integers, which are printed in full. Every such cycle with cycle multiplier L_q has $q \equiv 4$ or $20 \pmod{24}$, a short segment of length 1, 2, 5 or 6, all four of which occur, a long multiplier composed of primitive prime divisors of F_{2q}, and states above 10^{78}. Computer-assisted parts, namely a residue covering modulo 2,3,5,7 and exhaustive searches over 6,249,456 patterns and $1.6 \cdot 10^9$ congruences, show that the list of cycles with cycle multiplier L_q is complete for $q \leq 547$ and that no two-amplification cycle has period at most 36.

Keywords: GCD-Augmented Fibonacci Recurrence; Cofactor Map; Cycles; Amplifications; Lucas Numbers; Golden Integers; Primitive Prime Divisors; Collatz-type Cycles

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1. Introduction

The recurrence $C(n) = C(n-1) + C(n-2) + \gcd\big(C(n-1), C(n-2)\big)$ [1] grows like a Fibonacci sequence, but its gcd changes at isolated steps. The author's earlier work gave closed forms for special seeds [2], factored the dynamics through a cofactor map R on coprime pairs

[3], and showed that infinitely many seeds never lock into a cycle of R, where the sequence becomes geometric [4]. Cycles are naturally classified by their number of *amplifications*, the steps per period at which the gcd grows. There are exactly three cycles with one amplification [5, Theorem 10]; cycles with 6 and 19 amplifications are known; and two amplifications were left open, since they seemed to need a new bound, the analogue of the m-cycle problem for $3x + 1$ [5, Section 6].

Recurrences that feed a greatest common divisor back into the sequence are easy to state and hard to analyse. Rowland's recurrence $a(n) = a(n-1) + \gcd\big(n, a(n-1)\big)$ with $a(1) = 7$ has increments that are all 1 or prime [6]: as here, long runs of steps with gcd 1 are interrupted by isolated steps at which the gcd carries arithmetic information. The EKG sequence, in which $a(n)$ is the least unused integer with $\gcd\big(a(n-1), a(n)\big) > 1$, is proved to be a permutation of the positive integers with linear growth, while its finer asymptotics remain conjectural [7]. The recurrence studied here differs from both in that the gcd perturbs a second-order linear recurrence. Between two amplifications the dynamics is multiplication by the unit φ of $\mathbb{Z}[\varphi]$, and the long-run question concerns the cycles of a map on coprime pairs rather than growth rates.

The analogy with the $3x + 1$ problem is precise. A cycle of the $3x + 1$ map with m local minima, an m-cycle, solves one linear equation per pattern [8]. The trivial cycle is the only 1-cycle [9], there are no 2-cycles [10], and no m-cycles for $m \leq 68$ [11] and $m \leq 91$ [12]. All of these results rest on lower bounds for linear forms in logarithms (see also [13]). For a second-order Collatz recurrence only 1-cycles are understood [14]; for Conway's subprime Fibonacci sequences, cycles with two runs are known only by computation [15]; and periodic integer solutions of conditional-division recurrences are built from Fibonacci–Lucas identities [16]. One would expect two-amplification cycles to be absent.

They exist. The smallest known, of period 188, has step multipliers 18049 and $L_{188}/18049$, and its states have 79 digits or more. The reason is structural. Written in $\mathbb{Z}[\varphi]$, the cycle equation has determinant $G - \varphi^q$, which is a unit whenever the cycle multiplier G equals the Lucas number L_q. Then, for even q, every factorization of L_q gives an admissible candidate, and only finitely many content conditions decide. The $3x + 1$ problem has no such mechanism (Section 7).

Our contributions are as follows.

1. A cycle criterion in $\mathbb{Z}[\varphi]$ for two amplifications (Lemmas 1–4, Proposition 5). This restates the cycle equation of [5, Proposition 13] in golden coordinates; we claim no novelty for it.
2. Twenty-five explicit cycles with exactly two amplifications, at ten periods, each determined by three integers (Theorem 7, Table 1); all triples are printed in full in Appendix A.
3. Structure of all such cycles with $G = L_q$ (Proposition 8, Theorem 9, Lemma 10, Corollary 12). This is partly computer-assisted: a residue covering modulo $2,3,5,7$.

4. A census, computer-assisted: for $G = L_q$ the list is complete for $q \le 547$, and 188 is the least period (Theorem 11). The factorizations of L_q behind it are archived, with every prime factor proved prime.
5. For $G > L_q$: an explicit per-period bound, the exclusion of an infinite family, and an exhaustive search showing that no two-amplification cycle has period at most 36 (Lemma 13, Proposition 14, Theorem 15).

To the author's knowledge, these are the first known cycles of R with exactly two amplifications; the cycles known before have 1, 6 or 19 amplifications [5].

2. Preliminaries

Let $C(1), C(2)$ be coprime positive integers (assumption A1; by the scaling theorem [2, Theorem 4] this loses no generality) and $C(n) = C(n-1) + C(n-2) + \gcd\big(C(n-1), C(n-2)\big)$. Writing $d_n = \gcd\big(C(n-1), C(n)\big)$, $C(n-1) = d_n\alpha_n$ and $C(n) = d_n\beta_n$, the *state* $\sigma_n = (\alpha_n, \beta_n)$ is a coprime pair of positive integers, and $\sigma_{n+1} = R(\sigma_n)$, $d_{n+1} = g\, d_n$, where

$$R(\alpha,\beta) = \left(\frac{\beta}{g}, \frac{\alpha+\beta+1}{g}\right), \qquad g = \gcd(\beta, \alpha+1)$$

[3, Theorem 5 and Lemma 4]. The integer g is the *step multiplier*; a step with $g > 1$ is an *amplification*. A *cycle* of R of least period q has *cycle multiplier* G, the product of its q step multipliers; along it the sequence locks in, $C(n+q) = G\, C(n)$. The facts of [2, 3] used here are not among the items corrected in their versions of 21 October 2026.

Let F_n, L_n be the Fibonacci and Lucas numbers and $\varphi, \psi = \big(1 \pm \sqrt{5}\big)/2$, so $\varphi^n = F_{n-1} + F_n\varphi$ and $\varphi\psi = -1$. For $w = a + b\varphi \in \mathbb{Z}[\varphi]$ we write $w' = a + b\psi$, $N(w) = ww' = a^2 + ab - b^2$ and $c(w) = \gcd(a, b)$, the *content*. Two facts are used throughout: $w \in m\mathbb{Z}[\varphi]$ if and only if $m \mid c(w)$, and $c(\eta w) = c(w)$ for every unit $\eta = \pm\varphi^k$. All Fibonacci–Lucas identities below follow from Binet's formula [17] and were checked symbolically. For a prime r, $\pi(r)$ is the order of φ in $(\mathbb{Z}[\varphi]/r)^\times$, the Pisano period; $\pi(2), \pi(3), \pi(5), \pi(7) = 3, 8, 20, 16$, and for $r \ne 2, 5$, $\pi(r) \mid r-1$ or $\pi(r) \mid 2(r+1)$ according as $r \equiv \pm 1$ or $\pm 2 \pmod 5$ [18]. A *primitive prime divisor* of F_n is a prime dividing F_n but no F_m with $1 \le m < n$ [19].

Results used from [5]. Three results of that paper are used below; we restate them so that the present paper can be read on its own.

(R1) [5, Lemma 2]. If $(\alpha', \beta') = R(\alpha, \beta)$ for a state (α, β) with step multiplier g, then $\beta' - \alpha' = (\alpha+1)/g \ge 1$. Hence $\beta > \alpha$ on every state of a cycle.

(R2) [5, Proposition 13]. Let a cycle of period q have $k \ge 1$ amplifications. Counting from a state (α, β) that follows an amplification, let the i-th amplification end a segment of $q_i \ge 1$ steps and have step multiplier $g_i \ge 2$, and put $G = g_1 \cdots g_k$ and $s_i = q_{i+1} + \cdots + q_k$. Then the pair $u = (\alpha+1, \beta+1)$, written as a column, satisfies the *cycle equation*

$$(G\, I - Q^q)\, u = \sum_{i=1}^{k} g_1 \cdots g_{i-1}(g_i - 1)\, \big(F_{s_i+1}, F_{s_i+2}\big),$$

where I is the identity matrix, Q is the 2×2 matrix with rows $(0,1)$ and $(1,1)$, and $\det(G\,I - Q^q) = G^2 - L_q G + (-1)^q \neq 0$. Consequently each tuple $(q_1, \ldots, q_k; g_1, \ldots, g_k)$ belongs to at most one cycle, and $G > \varphi^q$.

(R3) [5, Corollary 11(b)]. The only cycles of period at most 2 are the fixed point $(1,2)$ and the 2-cycle $(3,5) \to (5,9)$; each has one amplification per period.

Definition 1. Let a cycle of R have exactly two amplifications per period q, and let σ be the state just after one of them. Counting steps from 0 at σ, the amplifications are steps $q_1 - 1$ and $q - 1$, with $q_1, q_2 \geq 1$, $q_1 + q_2 = q$, and multipliers $g_1, g_2 \geq 2$. The tuple $(q_1, q_2; g_1, g_2)$ is the *pattern*; steps $0, \ldots, q_1 - 1$ form *segment 1* and the remaining steps *segment 2*. Put $G = g_1 g_2$, $t = G - L_q$ and $D = G^2 - L_q G + (-1)^q$. Starting after the other amplification gives the *rotation* $(q_2, q_1; g_2, g_1)$.

Each pattern carries at most one cycle, and $G > \varphi^q$, by (R2). Since $\varphi^q = L_q - \psi^q$, this gives $t \geq 0$ for even q and $t \geq 1$ for odd q. Period $q = 2$ is excluded by (R3). The notation is collected in Table B1 (Appendix B).

Scope: S1, only the two-term recurrence is treated; S2, cycles with three or more amplifications are not classified; S3, cycles with $G > L_q$ are excluded only for the families of Propositions 5 and 14 and for $q \leq 36$; S4, the list of cycles with $G = L_q$ is complete only for $q \leq 547$ and for nine further periods (Theorem 11).

3. Golden Form and Cycle Criterion

To a pair (α, β) of positive integers we attach its *golden coordinate* $Y = (2\alpha - \beta + 1) + (\beta - \alpha)\varphi$; this is z/φ for the golden integer z of the shifted state in [4, Lemma 1]. Conversely $Y = a + b\varphi$ belongs to the pair $(a + b - 1,\ a + 2b - 1)$.

Lemma 1. For a pair (α, β) with golden coordinate Y, $\gcd(\alpha, \beta) = c(Y - 1)$. If (α, β) is a state, its step multiplier is $g = c(\varphi Y - 1)$ and the golden coordinate of $R(\alpha, \beta)$ is $1 + (\varphi Y - 1)/g$. In particular a step that does not amplify multiplies Y by φ.

Proof. $Y - 1 = (2\alpha - \beta) + (\beta - \alpha)\varphi$ has content $\gcd(\alpha, \beta - \alpha) = \gcd(\alpha, \beta)$. Next, $\varphi Y - 1 = (\beta - \alpha - 1) + (\alpha + 1)\varphi$ has content $\gcd(\beta, \alpha + 1) = g$, and a direct computation shows that $(\beta/g, (\alpha + \beta + 1)/g)$ has golden coordinate $1 + (\beta - \alpha - 1)/g + \big((\alpha + 1)/g\big)\varphi$. ■

Lemma 2. Let a cycle have pattern $(q_1, q_2; g_1, g_2)$, and let Y and X be the golden coordinates of the states after the second and after the first amplification. Then $\varphi^{q_1} Y - 1 = g_1(X - 1)$ and $\varphi^{q_2} X - 1 = g_2(Y - 1)$, so that

$$X = \frac{(g_1 - 1)g_2 + (g_2 - 1)\varphi^{q_1}}{G - \varphi^q}, \qquad Y = \frac{g_1(g_2 - 1) + (g_1 - 1)\varphi^{q_2}}{G - \varphi^q}.$$

Moreover $0 < X', Y' < 1$, and $D = N(G - \varphi^q) > 0$.

Proof. By Lemma 1, the $q_1 - 1$ non-amplifying steps multiply Y by φ, and the amplifying step gives $X = 1 + (\varphi^{q_1}Y - 1)/g_1$; segment 2 is the same. The system has determinant $G - \varphi^q \neq 0$, so Cramer's rule applies. For the conjugates, $|\psi^k| < 0.62$ for $k \geq 1$ and $|\psi^q| < 0.39$ for $q \geq 2$, so the numerator of X' exceeds $(g_1 - 1)g_2 - 0.62(g_2 - 1) > 0$, its denominator exceeds 3.6, and $X' < 1$ is equivalent to $(g_2 - 1)\psi^{q_1} + \psi^q < g_2$, which holds since the left side is below $0.62(g_2 - 1) + 0.39$; Y' is symmetric. Since $Y > 0$ on a cycle, $G - \varphi^q > 0$, hence $D = (G - \varphi^q)(G - \psi^q) > 0$. ■

Lemma 3 (cycle criterion). The pattern $(q_1, q_2; g_1, g_2)$ carries a cycle with exactly two amplifications per period if and only if (i) $X, Y \in \mathbb{Z}[\varphi]$; (ii) the pair of Y is positive; (iii) $c(\varphi^i Y - 1) = 1$ for $0 \leq i < q_1$ and $c(\varphi^i X - 1) = 1$ for $0 \leq i < q_2$; and (iv) $X \neq Y$. The step multipliers are then exactly g_1, g_2, and (iv) can fail only when $q_1 = q_2$ and $g_1 = g_2$.

Proof. Necessity: X and Y are golden coordinates of states; condition (iii) at $i = 0$ is the coprimality of the start state, and for $i \geq 1$ it says that the steps before each amplification do not amplify (Lemma 1); if $X = Y$, the least period divides $q_1 < q$. Sufficiency: by (i)–(iii) and Lemma 1, the steps $0, \ldots, q_1 - 2$ have multiplier 1, and step $q_1 - 1$ has multiplier $c(\varphi^{q_1}Y - 1) = c(g_1(X - 1)) = g_1 c(X - 1) = g_1$; segment 2 is the same, so the orbit returns after q steps with exactly two amplifications. A smaller period q' would divide q and repeat the multipliers, forcing $q = 2q'$, $q_1 = q_2$, $g_1 = g_2$ and $X = Y$. If $X = Y$, the first amplifications after X and after Y coincide, so $q_1 = q_2$ and $g_1 = g_2$. ■

Lemma 4 (local obstruction). If $c(\varphi^i W - 1) = 1$ for $0 \leq i < \lambda$, then for every prime r, $\varphi^i W \not\equiv 1 \pmod r$ for $0 \leq i < \lambda$; if $\lambda \geq \pi(r)$, then $W \bmod r$ is not a power of φ. For $r = 2,3$, where $\mathbb{Z}[\varphi]/r$ is the field with 4 or 9 elements and φ generates its multiplicative group, this means $W \equiv 0 \pmod r$.

Proof. The first claim is the first fact of Section 2; if $W \equiv \varphi^{-m}$ with $0 \leq m < \pi(r) \leq \lambda$, then $\varphi^m W \equiv 1$. The primes 2,3 are full in the sense of [4, Proposition 4]. ■

For a pattern with $X, Y \in \mathbb{Z}[\varphi]$ and a prime r, we call the conditions $\varphi^i Y \not\equiv 1$ for $0 \leq i < q_1$ and $\varphi^i X \not\equiv 1$ for $0 \leq i < q_2 \pmod r$ the *local conditions modulo* r. By Lemma 3(iii) and Lemma 4 they are necessary for a cycle, for every prime r. A pattern that satisfies the local conditions modulo each of 2,3,5,7 is a *local survivor*.

Proposition 5. A symmetric pattern $(m, m; g, g)$ gives $X = Y = (g - 1)/(g - \varphi^m)$, the golden coordinate of a single-amplification candidate of period m; it is integral only for $g = L_m$ (m even) or $g = L_m + 1$ (m odd) [5, Theorem 7]. Hence symmetric patterns carry no cycle with exactly two amplifications.

Proof. The numerators in Lemma 2 both equal $(g - 1)(g + \varphi^m)$, and $G - \varphi^{2m} = (g - \varphi^m)(g + \varphi^m)$; then (iv) of Lemma 3 fails. ■

4. Cycles with Cycle Multiplier L_q

Lemma 6. $D = 1$ if and only if q is even and $G = L_q$. In that case $G - \varphi^q = \varphi^{-q}$ is a unit, and $X = (L_q - g_2)\varphi^q + (g_2 - 1)\varphi^{q+q_1}, \qquad Y = (L_q - g_1)\varphi^q + (g_1 - 1)\varphi^{q+q_2}$.
So every pattern with $G = L_q$ satisfies (i) and (ii) of Lemma 3, and only the content conditions (iii) decide.

Proof. $D = Gt + (-1)^q$ with $G \geq 4$, and $t \geq 1$ for odd q, so $D = 1$ forces q even and $t = 0$; then $G - \varphi^q = \psi^q = \varphi^{-q}$, and Lemma 2 gives the formulas. Positivity: by $\varphi^q L_q = \varphi^{2q} + 1$, the pair of Y is $(F_{2q+1}, F_{2q+2}) - g_1(F_{q+1}, F_{q+2}) + (g_1 - 1)(F_{q+q_2+1}, F_{q+q_2+2})$, and $g_1 \leq L_q/2$ with $L_q F_{q+k} = F_{2q+k} + 1$ ($k = 1,2$; q even) makes both entries positive. ■

For $q_1 \neq q_2$ we call the shorter segment, of length s, *short* and the other, of length $\ell = q - s$, *long*. The step multiplier ending the long segment is the *long multiplier* h, and the one ending the short segment is the *short multiplier* $\hat{h} = L_q/h$.

Theorem 7 (existence). For each triple (q, s, h) of Table 1, the state
$\sigma^*(q, s, h) = (F_{2q+1}, F_{2q+2}) + (h - 1)(F_{q+s+1}, F_{q+s+2}) - h(F_{q+1}, F_{q+2})$
lies on a cycle of R of least period q with exactly two amplifications per period: counted from σ^*, step $q - s - 1$ has multiplier h and step $q - 1$ has multiplier $\hat{h} = L_q/h$, so the cycle multiplier is L_q. The smallest example has $q = 188$, $s = 2$ and the prime $h = 18049$, where $L_{188} = 7 \cdot 18049 \cdot 100769 \cdot 153037630649666194962091443041$; its state $\sigma^* = (\alpha^*, \beta^*)$ is

α* = 2746979206 9499419831 8230287562 8764160341 8162117658 1521798472 4294808180 616881803,
β* = 4444705723 2342374988 3397351998 2908586547 8883623794 7355748078 8128453000 056228601.

Proof. A finite exact computation. For the pattern $(s, q - s; \hat{h}, h)$, Lemma 6 gives X, Y; every content in (iii) of Lemma 3 equals 1, and $s < q/2$ gives (iv). The state of X is σ^* (Lemma 10). Direct iteration of R from σ^* and the integer cycle equation (R2) confirm each cycle independently. ■

Table 1. The twenty-five known cycles with exactly two amplifications ($G = L_q$). Each is reconstructed from (q, s, h) by Theorem 7. Values of h with more than 40 digits are abbreviated here by their number of digits; Table A1 (Appendix A) prints h and $\hat{h} = L_q/h$ in full for all rows. The first column numbers the rows. Rows 1–13 are in the order of the first version of this paper, and rows 14–25 were added in its revision; this order is also that of the archived data. Rows 1–10 are all such cycles with $q \leq 547$ (Theorem 11).

#	q	s	long multiplier h	h prime	digits of L_q/h	digits of β^*
1	188	2	18049	yes	36	79
2	236	5	392995850405749916464295600624151 43	no	15	99
3	236	6	28713074479853139217088887310 89	yes	19	99
4	244	5	25553609562771866971783	no	29	103
5	244	6	52471477541626010209	yes	32	103

6	268	2	27394199849089	no	43	113
7	364	5	1607900243355223	no	61	153
8	364	5	112739025441223	no	63	153
9	364	5	(58 digits)	no	19	153
10	532	5	(85 digits)	no	27	223
11	692	2	(53 digits)	no	93	290
12	692	6	(42 digits)	no	104	290
13	692	6	(103 digits)	no	43	290
14	556	1	(45 digits)	no	72	233
15	604	2	1146134421052077514508781114144769	no	94	253
16	652	1	(85 digits)	no	52	273
17	652	1	(96 digits)	no	41	273
18	652	2	152218087412280529	no	120	273
19	652	2	(70 digits)	no	67	273
20	652	2	(81 digits)	no	56	273
21	652	2	(83 digits)	no	55	273
22	652	2	(85 digits)	no	52	273
23	652	6	(81 digits)	no	56	273
24	652	6	(85 digits)	no	52	273
25	652	6	(119 digits)	no	19	273

Proposition 8 (multipliers). Let a cycle with exactly two amplifications have $G = L_q$, and let μ be the multiplier ending a segment of length λ. (a) If $\lambda \geq 3$, then μ is odd. (b) Every odd prime $r \mid \mu$ satisfies $\pi(r) \mid 4q$, $\pi(r) \nmid 2q$ and $\pi(r) > \lambda$. If the segment is the long one, $5s \leq q$ and $3 \nmid q$, then $\pi(r) = 4q$: r is a primitive prime divisor of F_{2q}, and $r \equiv 1 \pmod{4q}$ or $r \equiv -1 \pmod{2q}$. (c) If $q \equiv 4 \pmod 8$ and $\ell \geq 16$, then $7 \mid L_q/h$.

Proof. Count the pattern so that the segment is segment 2. For $r \mid \mu$, $r \mid L_q$ and Lemma 6 gives $X \equiv -\varphi^{q+q_1}$, so Lemma 4 yields $-\varphi^{q+q_1+i} \not\equiv 1 \pmod r$ for $0 \leq i < \lambda$. (a) For $r = 2$ this excludes three consecutive powers of φ, impossible as $\pi(2) = 3$. (b) From $\varphi^q \equiv -\psi^q$ we get $\varphi^{2q} \equiv -1$, hence $\pi(r) \mid 4q$ and $\pi(r) \nmid 2q$; writing $-1 = \varphi^{2q}$, no multiple of $\pi(r)$ lies in $[4q - \lambda, 4q - 1]$, i.e. $\pi(r) > \lambda$. With $\pi(r) = 4q/d$, d odd, $d = 3$ needs $3 \mid q$ and $d \geq 5$ gives $\pi(r) \leq 4q/5 \leq \ell$. Finally $\pi(r) = 4q$ holds exactly when the least n with $r \mid F_n$ is $2q$: if the least such n is smaller, it divides $2q$ and differs from q (as $\gcd(F_q, L_q) \mid 2$), so $n \leq 2q/3$; then φ^n lies in the prime field, so φ^{2n} equals its norm $(-1)^n$ and $\pi(r) \mid 4n < 4q$; conversely $\varphi^{4q/d} = 1$ with $d > 1$ gives $\varphi^{2q/d} \equiv \pm 1$ (also when r splits, by the norm). The congruences follow from [18]. (c) $7 \mid L_n$ exactly for $n \equiv 4 \pmod 8$, and $\pi(7) = 16 \leq \ell$ excludes $7 \mid h$ by (b). ■

Theorem 9 (residue covering). Let a cycle with exactly two amplifications have $G = L_q$ and a long segment of length at least 20. Then $s \in \{1,2,5,6\}$ and $q \equiv 4$ or $20 \pmod{24}$. Moreover, $6 \nmid q$ for every such cycle, with no condition on the lengths.

Proof. Fix $r \in \{2,3,5,7\}$; all four Pisano periods divide 240. By Lemma 6, X and Y modulo r depend only on $q, q_1, q_2 \bmod \pi(r)$ and on $(g_1, g_2) \bmod r$ with $g_1 g_2 \equiv L_q$, and the local

conditions modulo r depend only on the first $\min(\ell, \pi(r))$ values of i. Hence they depend only on the class of (s, ℓ), $s \le 19$, or of (q_1, q_2) when both lengths are at least 20, modulo 240. A class is excluded once one prime admits no residue pair; this is a superset argument. We enumerated all 31,080 classes with two independent programs (golden residues; integer states from the cycle equation (R2), iterated modulo r), which agree: exactly the classes with $s \in \{1,2,5,6\}$ and $q \equiv 4,20 \pmod{24}$ survive. For $6 \mid q$ by hand: $L_q \equiv 0$ and $\varphi^q \equiv 1 \pmod 2$, so $Y \equiv g_1 + (g_1 + 1)\varphi^{q_2} \not\equiv 0$, and likewise X; but a segment of length at least 3 needs $W \equiv 0 \pmod 2$ by Lemma 4. ■

We call a period q *admissible* if $q \equiv 4$ or $20 \pmod{24}$. By Theorem 9, a cycle with $G = L_q$ whose long segment has length at least 20 has an admissible period; by Corollary 12 below, this covers every cycle with $G = L_q$. There are 30 admissible periods with $360 < q \le 716$.

Lemma 10 (anchor states). Let $s < q/2$ and $G = L_q$. The states after the amplifications ending the short and the long segment are $\sigma^*(q, s, h)$ and
$\sigma^\circ = (F_{2q+1}, F_{2q+2}) + (\hat{h} - 1)(F_{2q-s+1}, F_{2q-s+2}) - \hat{h}(F_{q+1}, F_{q+2}), \qquad \hat{h} = L_q/h.$
Every state (α, β) of the cycle has $\alpha > F_{2q+1}$ and $\beta > F_{2q+2}$, and the cycle is determined by (q, s, h).

Proof. By Lemma 6 and $\varphi^q L_q = \varphi^{2q} + 1$, $X = \varphi^{2q} + 1 - h\varphi^q + (h-1)\varphi^{q+s}$, and the pair of $a + b\varphi$ is the image of $a + b\varphi$ under the linear map with $\varphi^n \mapsto (F_{n+1}, F_{n+2})$, $1 \mapsto (1,1)$, minus $(1,1)$; Y is analogous. Non-amplifying steps $(\alpha, \beta) \mapsto (\beta, \alpha + \beta + 1)$ increase both entries, so their minima occur at σ^* or σ°. With $h \ge 3$ (Proposition 8(a)) and $k = 1,2$, entry k of σ^* exceeds F_{2q+k} by $h(F_{q+s+k} - F_{q+k}) - F_{q+s+k} \ge F_{q+k-3} > 0$, and that of σ° by at least $(\hat{h} - 1)F_{q+k+1} - F_{q+k} > 0$. A pattern carries at most one cycle. ■

Theorem 11 (census). (a) The cycles with exactly two amplifications and $G = L_q$ of period $q \le 547$ are exactly rows 1–10 of Table 1. In particular, 188 is the least period when $G = L_q$. (b) Those of period $q \in \{556,572,580,644,652,676,692,700,716\}$ are exactly rows 11–14 and 16–25 of Table 1. (c) At each of the six remaining admissible periods up to 716, $q \in \{548,596,604,620,628,668\}$, every such cycle other than row 15 has a long multiplier divisible by a prime factor of a composite cofactor of L_q that is not yet factored.

Proof. For even $q \le 360$ the content test of Lemma 3 was run on every factorization $L_q = g_1 g_2$ and every q_1, one rotation per cycle: 6,249,456 patterns, six of which pass. Each L_q was factored completely with PARI/GP 2.17.2 [20] (function factor). The factorizations were computed for this work and were not taken from tables; the classical published tables are [21]. Every prime factor was then proved prime with the PARI function isprime in its default setting, which applies a Baillie–Pomerance–Selfridge–Wagstaff test and then gives a proof by the $p - 1$ test or by the APR-CL test: 434 distinct primes in all, the largest with 69 digits. The pattern counts match the divisor counts; direct iteration of R agrees with the content test on 74,773 patterns, including all whose first failing content has no prime factor up to 47.

For $360 < q \leq 716$ the long segment has length at least 20, so Theorem 9 and Proposition 8 restrict the search to the 30 admissible q, to $s \in \{1,2,5,6\}$ and to long multipliers h that are products of primitive prime divisors of F_{2q} dividing L_q. With a limit of 90 seconds per period, PARI factored L_q completely at 16 of these 30 periods. At each of the other 14 we removed from L_q the primes of L_{4d} for the proper divisors d of $q/4$, which are not primitive; divided the cofactor by all primes $r < 10^9$ with $r \equiv -1 \pmod{2q}$ or $r \equiv 1 \pmod{4q}$ (Proposition 8(b)); and ran the elliptic curve method (GMP-ECM 7.0.5 [22]) for about 200 seconds per period, with first-stage bounds increasing from $2 \cdot 10^3$ to at most $3 \cdot 10^6$. The composite cofactors of at most 84 digits that remained, at $q = 452$, 604 and 652, were then given to PARI's factor for up to 25 minutes each; this finished at $q = 452$ and $q = 652$. This completed eight further periods, so that L_q is completely factored at 24 of the 30 admissible periods; composite cofactors of 84 to 120 digits remain at the six periods of (c). Again every prime was proved prime. All pairs (h, s) with h built from the known primitive primes were tested with Lemma 3: 1,368 pairs at the completely factored periods and 120 at those of (c). Exactly 19 pass, rows 7–25 of Table 1. At a completely factored period every primitive prime is known, which, with the first part of the proof, gives (a) and (b); at a period of (c), a long multiplier that has not been tested contains a prime of an unfactored cofactor, which gives (c).

All factorizations used, complete and partial, are archived with the code [23] in the file lucas_factorizations.csv; the script re-proves the primality of every listed prime (part factorizations) and repeats the search above period 360 from the archived factorizations, without any factoring (part further). Appendix A prints the factorizations of L_q at the periods of Table 1. ■

Corollary 12 (structure). Every cycle with exactly two amplifications and $G = L_q$ satisfies: (a) $q \equiv 4$ or $20 \pmod{24}$; (b) $s \in \{1,2,5,6\}$; (c) every prime factor of h is a primitive prime divisor of F_{2q}, and $7 \mid L_q/h$; (d) it passes through $\sigma^*(q, s, h)$, and both entries of every state exceed $F_{2q+1} \geq F_{377} > 2.7 \cdot 10^{78}$.

Proof. Only the exhaustive part of Theorem 11, for even $q \leq 360$, is used; that part relies on the content test alone and not on Theorem 9 or Proposition 8, so the argument is not circular. It shows that $q \geq 188$. Hence $\ell = q - s \geq q/2 \geq 94$, and Theorem 9 gives (a) and (b). Then $5s \leq 30 \leq q$, $3 \nmid q$ and $q \equiv 4 \pmod{8}$, so Proposition 8(b) and (c) give (c), and Lemma 10 gives (d). ■

5. Patterns with $G > L_q$

Now $t \geq 1$, so $D = Gt + (-1)^q > 1$, $G - \varphi^q$ is not a unit, and integrality is a genuine restriction.

Lemma 13. Let $(q_1, q_2; g_1, g_2)$ be a pattern with $t \geq 1$, put $\tau = G - \varphi^q = t + \psi^q > 0$ and $K = F_{q+q_2} + tF_{q_2}$, and let $Y = a + b\varphi$. (a) $b = N_b/D$ with $N_b = g_1(K - F_q) + F_q G - K$; for fixed q, q_1, t, the condition $D \mid N_b$ is a linear congruence in g_1. (b) $b\tau = (g_1 - 1)F_{q_2} + Y'F_q$, and symmetrically for X. (c) On a cycle, $t \leq F_q + 1$ or $(t - 1 - F_q)^2 < (L_q + t)F_{q_1}F_{q_2}$; in particular $t \leq 3F_q$.

Proof. Taking coordinates in $Y(G-\varphi^q) = (G-g_1) + (g_1-1)\varphi^{q_2}$ gives a 2×2 integer system with determinant D; Cramer's rule and the addition formula $F_{q_2}F_{q+1} + F_{q_2-1}F_q = F_{q+q_2}$ give (a). Its second equation reads $b\big(F_{q-1}+t\big) - aF_q = (g_1-1)F_{q_2}$; substituting $a = b/\varphi + Y'$ and $F_{q-1} - F_q/\varphi = \psi^q$ gives (b). On a cycle $b = \beta - \alpha \geq 1$ by (R1) and $0 < Y' < 1$ (Lemma 2), so (b) yields $(g_1-1)F_{q_2} > \tau - F_q$, and likewise $(g_2-1)F_{q_1} > \tau - F_q$. If $t > F_q + 1$, then $\tau - F_q > t - 1 - F_q > 0$; multiplying and using $(g_1-1)(g_2-1) < G = L_q + t$ gives (c). Since $5F_{q_1}F_{q_2} = L_q - (-1)^{q_2}L_{q_1-q_2} \leq L_q + L_{q-2} = 5F_{q-1}$, the inequality in (c) fails at $t = 3F_q$ for $q \geq 5$, and for all larger t, since the difference of its sides is convex in t and increasing from there. For $q = 3$ and $q = 4$ the largest value of $F_{q_1}F_{q_2}$ is 1 and 2, so the inequality implies $(t-3)^2 < t+4$ and $(t-4)^2 < 2(t+7)$, which fail for $t > 6 = 3F_3$ and for $t > 9 = 3F_4$, respectively. ■

So every period admits only finitely many candidates, and they can be enumerated: for each q_1 and t, solve the congruence of Lemma 13(a), keep the solutions g_1 dividing G, and test Lemma 3.

Proposition 14 (the $t = 2$ family). Let m be even, $n = L_m$ and $q = 3m$. The pattern $\big(m, 2m;\ n-1, (n-1)(n+2)\big)$ has $G = L_{3m} + 2$ and, with $x = \varphi^m$,
$Y = (2-n) + (n^2-n-1)x, \qquad X = (1-n) + (n^2-2)x,$
so it is integral for every even m. It never carries a cycle: for $m \geq 6$ its start state is not coprime, because $c(Y-1)$ is divisible by $L_{m/2}$ ($m/2$ odd) or by $F_{m/2}$ ($m/2$ even); for $m = 2,4$ a step amplifies early.

Proof. For even m, $L_{3m} = n^3 - 3n$ and $(n-1)^2(n+2) = n^3 - 3n + 2$. In $\mathbb{Z}[x]$ we have $x^2 = nx - 1$, hence $x^3 = (n^2-1)x - n$ and $G - x^3 = (n^3 - 2n + 2) - (n^2-1)x$. Multiplying Y and X by this element and reducing x^2 gives $Y(G-x^3) = g_1(g_2-1) + (g_1-1)x^2$ and $X(G-x^3) = (g_1-1)g_2 + (g_2-1)x$, which are the numerators of Lemma 2 for $q_1 = m$, $q_2 = 2m$. Both are identities between polynomials in n and x modulo $x^2 - nx + 1$; they were checked symbolically, and the script [23] checks them numerically for even $m \leq 100$ (part t2family). Let $k = m/2$ and $p = L_k$ (k odd) or $p = F_k$ (k even). Modulo p we have $\varphi^k \equiv \mp\psi^k$, hence $x = \varphi^{2k} \equiv 1$, and $n = L_{2k} = L_k^2 + 2 = 5F_k^2 + 2$ in the respective cases, so $n \equiv 2$. Then $Y - 1 \equiv (2-2-1) + (4-2-1) = 0 \pmod p$, and $p \geq 3$ for $m \geq 6$; so (iii) of Lemma 3 fails at $i = 0$. For $m = 2$ the pattern is $(2,4;\ 2,10)$ with $X = 5 + 7\varphi$, and $c(\varphi X - 1) = c(6 + 12\varphi) = 6$; for $m = 4$ it is $(4,8;\ 6,54)$ with $Y = 77 + 123\varphi$, and $c(\varphi Y - 1) = c(122 + 200\varphi) = 2$. In both cases (iii) of Lemma 3 fails at $i = 1$. ■

Theorem 15. For $3 \leq q \leq 36$, the integral patterns with $G > L_q$ allowed by Lemma 13(c) are 26: seventeen symmetric patterns (Proposition 5), the six members of Proposition 14 with $q \leq 36$, and three sporadic candidates, $(2,3;\ 3,5)$, $(2,5;\ 8,4)$ and $(3,5;\ 13,4)$, of periods 5,7,8. None carries a cycle. Consequently, no cycle of R with exactly two amplifications has period $q \leq 36$.

Proof. Computer-assisted, by the enumeration after Lemma 13 ($1.6 \cdot 10^9$ congruences; part range of the script); a second program that tests every divisor of every $G \leq L_q + 3F_q$ gives the same candidates for $q \leq 20$. Each candidate fails Lemma 3 (start state not coprime, an early amplification, or a repeat of a single-amplification cycle). With Theorem 11 and the exclusion of $q = 2$ by (R3), the consequence follows. ■

6. Main Results and Computations

Corollary 16. (a) Cycles of R with exactly two amplifications per period exist; the least period of such a cycle with $G = L_q$ is 188, and no such cycle has period $q \leq 36$. (b) Cycles of R with exactly 1, 2, 6 and 19 amplifications per period exist; those with one amplification are exactly the three of [5, Theorem 10], and cycles with 6 and 19 amplifications occur at periods 22 and 65 [5, Section 5].

Boundary recovery. With one amplification per period, the golden form gives $X = (g-1)/(g-\varphi^q)$ and $N(X) = (g-1)^2/D$ with $G = g$ in D, which is the norm formula of [5, Lemma 5]. Integrality makes $N(X)$ a positive integer; $N(X) = 1$ forces $(L_q - 2)g = (-1)^q - 1$, i.e. $(q, g) = (1,2)$, and otherwise $(g-1)^2 \geq 2D$, hence $g \leq 2L_q + 2$; a full scan of this range for $q \leq 20$ finds X integral exactly for $g = L_q$ (q even) and $g = L_q + 1$ (q odd) [5, Theorem 7]. The content conditions then select exactly $(q, g) = (1,2), (2,3), (4,7)$, and the cycles contain the published states $(1,2)$; $(3,5), (5,9)$; and $(29,47), (125,203)$ [5, Theorem 10].

Computations. Figure 1 summarizes the census of Theorem 11 for $q \leq 360$. Of the 6,249,456 patterns, the first failing condition is a start state that is not coprime (4,565,005), an early amplification in segment 1 (1,631,723) or in segment 2 (14,427), or a larger multiplier at the end of segment 1 (38,295). The local conditions (Section 3) remove almost all patterns: the least prime among 2,3,5,7 whose local condition fails is 2 for 5,468,403 patterns, 3 for 703,945, 5 for 76,635 and 7 for 376, and only 97 patterns are local survivors. Except for one at $q = 10$ and one at $q = 28$, both with segments shorter than 20, the local survivors have $q \equiv 4,20 \pmod{24}$ and $s \in \{1,2,5,6\}$, as Theorem 9 predicts, and six of them are cycles. As a consistency check, iterating R from all 24,463 coprime seeds in $[1,200]^2$ finds exactly five cycles, with 1,1,1,6 and 19 amplifications and periods 1,2,4,22,65, as in [5, Section 5] and [4].

Software, run times and independent checks. All arithmetic is exact (Python 3.13, PARI/GP 2.17.2 [20]). A public script reproduces every number of this paper and reconstructs the cycles of Table 1 from their triples alone [23]. On one core of a virtual machine with two Intel Xeon cores at 2.1 GHz and 7 GB of memory, its parts take 0.4 seconds for the residue covering (Theorem 9), 3 minutes for the census up to $q = 360$ and 23 minutes for the restricted search up to $q = 716$ with the limit of 90 seconds per period (Theorem 11; 21 of these minutes are the limits at the 14 periods that this run does not complete; the additional factoring at those periods is described in the proof of Theorem 11), 58 minutes for the enumeration with $G > L_q$ (Theorem 15), and 70 seconds for the seed census. The computer-assisted statements can also be checked without this script. Theorem 7 needs only the iteration of R from the twenty-five states $\sigma^*(q, s, h)$, with the triples of Table A1. Theorem 9 is a finite enumeration of residues modulo 2,3,5,7. Theorems 11 and 15 are the enumerations described in their proofs, and the first of them starts from the archived factorizations, each of which is verified by multiplying its factors and proving them prime.

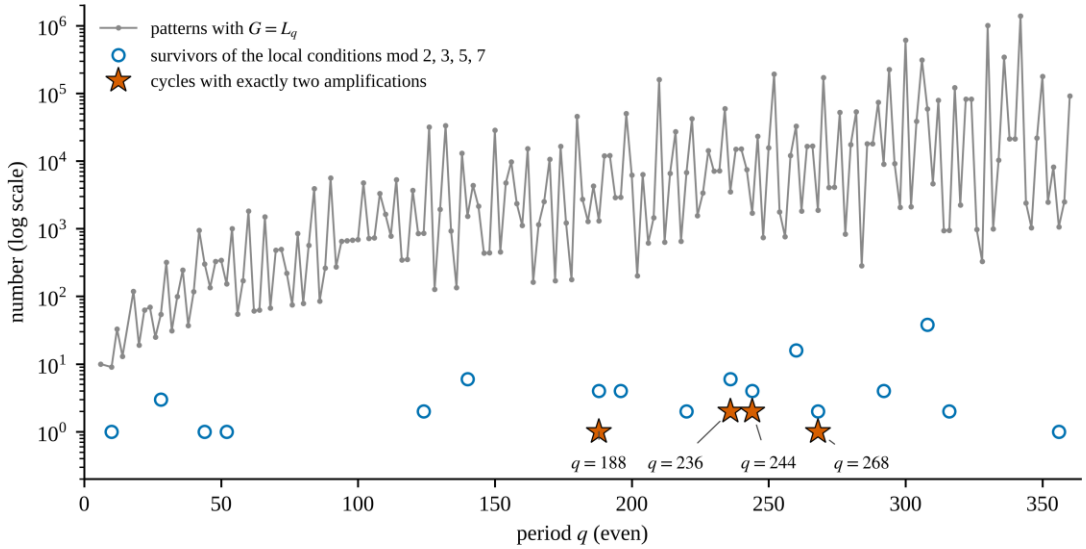


Figure 1. The family $G = L_q$ for even $q \leq 360$ (logarithmic scale): the number of patterns for each period (grey), the patterns that pass the local conditions modulo 2, 3, 5 and 7 (open circles) and the cycles with exactly two amplifications (stars, labelled with their periods). Local survivors with $q \geq 40$ occur only at $q \equiv 4{,}20 \pmod{24}$ (Theorem 9).

7. Discussion

Mechanism. For the $3x + 1$ map, a cycle with prescribed pattern solves one linear equation whose denominator is $2^K - 3^k$ [8]. That denominator equals 1 only for the trivial cycle: $2^K \equiv 1 \pmod 3$ forces K even, and then $\left(2^{K/2} - 1\right)\left(2^{K/2} + 1\right) = 3^k$ forces $K = 2$, $k = 1$. Here the corresponding determinant is $G - \varphi^q$, and Lemma 6 shows that it is a unit for *every* even q as soon as $G = L_q$, so every factorization of L_q is a candidate. Most fail at a small prime (Figure 1), but the number of factorizations grows with q, and some survive. The cycles are large: by Corollary 12 both entries of every state exceed F_{377}, which is why the censuses of [4, 5], with seeds up to 200, could not see them.

Does the mechanism transfer? It is not special to two amplifications. By (R2), the cycle equation of a cycle with any number $k \geq 2$ of amplifications has the same matrix $G\,I - Q^q$, whose determinant is D; so $D = 1$ exactly when q is even and $G = L_q$, and every factorization of L_q into k factors is then an integral candidate. Whether cycles with three or more amplifications arise in this way is part of S2. The same algebra applies to $C(n) = aC(n-1) + C(n-2) + \gcd\left(C(n-1), C(n-2)\right)$ with an integer $a \geq 2$. Its cofactor map is $(\alpha, \beta) \mapsto (\beta, \alpha + a\beta + 1)/\gcd(\beta, \alpha + 1)$, the matrix Q gets the rows $(0{,}1)$ and $(1, a)$, and $\det(G\,I - Q^q) = G^2 - \mathrm{tr}(Q^q)\,G + (-1)^q$ equals 1 when q is even and $G = \mathrm{tr}(Q^q)$; we have not examined the integrality and content conditions in that case. What the mechanism needs is a unit. It is therefore absent for first-order maps such as $3x + 1$, as shown above, and for the Tribonacci analogue of [5, Section 6] the determinant is a cubic polynomial in G, for which the question is open (S1).

Limitations and future work. Each scope label marks an open problem. S3: does every cycle with exactly two amplifications have $G = L_q$? Lemma 13 reduces the question to two linked

Diophantine conditions, and all integral candidates found for $q \leq 36$ have $t \leq 5$ or are symmetric. A proof would have to replace the bound $t \leq 3F_q$ of Lemma 13(c), which grows exponentially with q, by a bounded one. Transcendence methods do not help here: the linear form $G - \varphi^q = t + \psi^q$ is bounded below trivially, by $t - 0.39$, whereas for the $3x + 1$ problem the lower bound for $|2^K - 3^k|$ is the heart of the matter [11]. What is missing is an arithmetic argument that $D = Gt + (-1)^q$ cannot divide the Cramer numerators of both X and Y (Lemma 13(a)) unless the pattern is symmetric or t is very small. A positive answer would make 188 the least period overall. Theorem 9 allows the short lengths 1, 2, 5 and 6, and all four occur. Three cycles, one of period 556 and two of period 652 (rows 14 and 16–17 of Table 1), have $s = 1$, although all 157 restricted candidates with $s = 1$ and $q \leq 360$ fail. Both residue classes of q modulo 24 occur as well, so Theorem 9 cannot be sharpened in these respects. S4: completeness beyond the periods of Theorem 11 depends on factoring the primitive parts of L_q; composite cofactors of 84 to 120 digits remain at 6 admissible periods up to 716. S2 and S1: cycles with three or more amplifications, and the Tribonacci analogue with its cubic norm form, remain open.

Conjecture 1. There are infinitely many cycles of R with exactly two amplifications, already with $G = L_q$.

The evidence is the twenty-five cycles at ten periods up to 692, found although the search beyond 360 is complete for only 24 of the 30 admissible periods, and the fact that every admissible period offers candidates, since F_{2q} has a primitive prime divisor [19]. The additional factoring behind Theorem 11 was a small test of the conjecture: the first thirteen cycles were known when it was run at 14 further periods, it completed eight of them, and twelve more cycles appeared. The yield appears to grow with the number of primitive prime divisors: the period 652, at which L_q has seven of them, alone carries ten cycles.

A labelled data set for in-context prediction. The census is also a labelled data set: 6,249,456 patterns, each tagged with its first failing condition and with the least prime among 2,3,5,7 whose local condition fails (Section 6), and six of them tagged as cycles. Whether a learned predictor can anticipate these labels from the pattern alone is an open question, and the theory of in-context learning suggests where the difficulty should lie. Transformers can be trained to learn simple function classes from examples given in the prompt [24], and small transformers trained to compute greatest common divisors learn a list of divisors, built from the divisors of the number base and small primes, and predict the largest one that divides both inputs [25]; this is the kind of information that the local conditions carry. The framework of [26] makes the expectation precise. A function class is learnable in context, with the sample complexity of standard learning, when its sufficient statistic is a sum over the examples [26, Theorem 1]; it is not learnable by constant-depth transformers of polynomial size when the statistic requires identifying a discrete structure among exponentially many candidates, or a long sequential computation [26, Theorems 2 and 5]. Both regimes occur here. By the proof of Theorem 9, the local label depends only on (g_1, g_2) modulo 210, on (q_1, q_2) modulo 240 and on the exact value of a segment length below 20. It is a finite table. Once these residues are supplied as features, for instance to a transformer as learned categorical embeddings like the constellation embeddings of [27, Section 3.2], the table is estimated from label counts per class, an additive statistic. Delay estimation in GNSS-integrated sensing falls in the easy regime for the same reason, since its

Fisher information is additive, while fault detection and integrity monitoring fall in the hard one [28, Section 5]. The label "cycle" is of the other kind: by Proposition 8 it asks which products h of primitive prime divisors of F_{2q} pass q successive content conditions, that is, for a subset of the prime factors of L_q and a computation of length q.

This asymmetry has the same shape as those established for geospatial tasks within the same framework, and each of them suggests a concrete test on the census. Computing a cycle from its triple (q, s, h) is a forward problem with a closed form (Theorem 7), whereas finding all triples for a given q is an inverse problem that needs the factorization of L_q; for the gravity field, forward evaluation is the easy class and the localization of J discrete sources the hard one [29, Theorems 3.4 and 5.2]. The number of prime factors of h plays the role of the number J of instances in [30, Theorem 5.4 and Corollary 5.2], where localization becomes hard beyond a threshold of about 2 to 4 instances. In Table A1, the long multiplier h has between 1 and 6 prime factors, chosen among at most 7 primitive prime divisors of F_{2q}, so the present data straddle that threshold. Cycles are rare events, six among 6,249,456 patterns, so overall accuracy is uninformative and the task is one of detection, the regime analysed for extreme events in [31, Theorem 5.3]. Structural side information shrinks the search before any learning: Theorem 9 and Proposition 8 reduce the 6,249,456 patterns to 628 candidates, much as the specificity of a query reduces the effective number of objects in [32, Theorem 4.2]. Writing out the q steps of the orbit as intermediate tokens is the natural chain of thought, and [33, Theorems 6.1 and 6.2] predict that the number of such tokens must grow at least linearly with the number of elements to be identified. Finally, whatever a predictor proposes, a cycle is certified only by exact arithmetic. This is the division of labour between learned estimation and explicit verification for which [34, Theorem 12] argues in safety-critical positioning, and which [35, Section 4.4] states more generally: provable guarantees and learned components are complements, not alternatives. It is also the safe-fallback architecture proposed in [36, Section 6.5], in which a learned output is accepted only when it passes a check with a provable error bound; here the check is exact. The expectations above are predictions of a theoretical framework whose hardness results are worst-case statements [26, Section 6.4], and the chain-of-thought bounds have not been validated empirically [33, Section 7.4]. No learning experiment was run for this paper; the census is offered as a test bed. Because every pattern is labelled, a predictor can be scored on the whole population, with the base rate known and with misses and false alarms both counted, rather than on selected cases; [37, Sections 3.3 and 6.3] argues for this standard in the evaluation of AI systems.

8. Conclusions

Cycles of the cofactor map with exactly two amplifications per period exist. The smallest known has period 188; twenty-five are given explicitly, at ten periods, each by three integers. For cycle multiplier L_q, the determinant of the cycle equation in $\mathbb{Z}[\varphi]$ is a unit, which explains their existence. With $G = L_q$, their structure is rigid: $q \equiv 4{,}20 \pmod{24}$, a short segment of length 1, 2, 5 or 6, a long multiplier made of primitive prime divisors of F_{2q}, and states above 10^{78}; all four short lengths occur, and the list is complete up to period 547. No cycle with two amplifications has period at most 36. Unlike the $3x + 1$ map, which has no 2-cycles, the gcd-augmented Fibonacci recurrence has cycles with 1, 2, 6 and 19 amplifications per period.

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Code and data availability. A script that reproduces all computations of this paper is archived at Zenodo [23], together with the factorizations of the Lucas numbers used (file lucas_factorizations.csv) and the triples of Table A1 in machine-readable form (file table1_full.csv). The labels in the output of the script are those of this paper.

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Appendix A. The Twenty-Five Cycles in Full

Table A1 gives, for every row of Table 1, the triple (q, s, h) and the short multiplier $\hat{h} = L_q/h$ in full. Digits are printed in groups of ten, to be read without the spaces. By Theorem 7, each row determines its cycle: the state $\sigma^*(q, s, h)$ is computed from the triple and Fibonacci numbers, and q steps of R return to it, with multiplier h at step $q - s - 1$, multiplier $\hat{h}$ at step $q - 1$ and multiplier 1 at every other step. The column "primes of h" refers to the primitive prime divisors $p_1 < p_2 < \cdots$ of F_{2q} that are known to divide L_q, listed after the table. The same data are archived in machine-readable form [23].

Table A1. The twenty-five known cycles in full: period q, short length s, long multiplier h and short multiplier $\hat{h} = L_q/h$. The rows are those of Table 1, in the same order.

row	q	s	primes of h		value of h (first line) and of $\hat{h} = L_q/h$ (second line)
1	188	2	p_1	h	18049
				$\hat{h}$	1079501430 2055348960 0944948366 589703
2	236	5	p_2p_4	h	3929958504 0574991646 4295600624 15143
				$\hat{h}$	5329606367 98849
3	236	6	p_4	h	2871307447 9853139217 0888873108 9
				$\hat{h}$	7294632235 865846263
4	244	5	p_1p_2	h	2555360956 2771866971 783
				$\hat{h}$	3850631561 1025067059 838423329
5	244	6	p_2	h	5247147754 1626010209
				$\hat{h}$	1875257570 2569207658 1413121612 23
6	268	2	p_1p_2	h	2739419984 9089
				$\hat{h}$	3724165454 4207550292 7120356985 7180680747 223
7	364	5	p_2p_4	h	1607900243 355223
				$\hat{h}$	7332329111 8815498785 9551194824 2950159031 1583635684 2641686380 9
8	364	5	$p_1p_2p_3$	h	1127390254 41223
				$\hat{h}$	1045747354 7615079756 2126873104 1382626194 1326008424 5925455125 809
9	364	5	$p_2p_3p_4p_5$	h	2167970958 1200419648 7894109489 7400376601 0263888065 28443943
				$\hat{h}$	5438105025 898658449
10	532	5	$p_1p_2p_4$	h	1963935786 1709953081 7287458660 9385961733 0477124877 3806466983 4056498633 3693605652 30903
				$\hat{h}$	7732097632 9858806363 4687249
11	692	2	$p_1p_2p_3$	h	2573413934 3360815502 0092523324 7906394528 6304136739 969
				$\hat{h}$	1617846839 2075221907 7906183330 3364817955 5415997567 0799178137 1550112004 5825940065 6173423199 063
12	692	6	p_2p_3	h	3654185644 5012805310 3699216003 1506414167 69

row	q	s	primes of h		value of h (first line) and of $\hat{h} = L_q/h$ (second line)
				$\hat{h}$	1139348135 1729295335 2702796574 0133671139 2506871850 6490220592 6140543909 9234708785 3567612264 8830734556 5463
13	692	6	p_1p_4	h	1627640193 1041850478 9575423677 1619530198 9295531215 2128886560 8772205585 6049583979 0810874664 1186763652 209
				$\hat{h}$	2557929951 1508963717 2589451202 2054489917 383
14	556	1	$p_1p_2p_3$	h	2026362389 9680640432 2047320340 2398151861 69623
				$\hat{h}$	7769818300 1716164689 0678074428 9157767005 6037341882 2237544317 9743282190 09
15	604	2	$p_1p_2p_4$	h	1146134421 0520775145 0878111414 4769
				$\hat{h}$	1476723395 5172111103 6250761328 8721749385 8664623831 5432844927 9970165346 5651721928 0888848694 1303
16	652	1	$p_2p_4p_5p_7$	h	6862459992 3876466409 8518280173 2832691615 8913308316 0024954488 9505596968 4055235972 80183
				$\hat{h}$	2651316761 5392554288 1941451716 5552308828 3334887622 09
17	652	1	$p_1p_2p_3p_4p_5p_7$	h	7283707158 1177924108 6035320509 5437206768 5231836490 9874541933 8878073043 1763353810 0629222769 604183
				$\hat{h}$	2497980054 4193811013 2479134338 8187731020 9
18	652	2	p_3p_4	h	1522180874 12280529
				$\hat{h}$	1195295218 3619430364 2754374746 4732549176 4844608174 5110506591 3706737288 3541436450 8179841806 7894892143 1373570096 4966213543
19	652	2	p_5p_6	h	4682215972 4956491069 3163552555 9578959733 5291653451 3993518000 0639204609
				$\hat{h}$	3885885510 2132574314 6746441507 0361425817 9533432030 9084812030 3671383
20	652	2	$p_1p_3p_5p_6$	h	4969630428 8186164902 1823347258 7791666535 5166346161 8759325372 3048193523 7025385660 9
				$\hat{h}$	3661148542 8978144899 9648484540 4339865824 9488531569 747383
21	652	2	$p_1p_4p_5p_6$	h	1140076209 9581095348 0483922943 3070446314 2437505909 2263735352 6647747198 2500660997 249
				$\hat{h}$	1595906926 5973413821 3215241334 6264747978 8762494136 12503
22	652	2	$p_2p_6p_7$	h	9987854582 0619016616 3207525206 6156397151 4723064029 6982547272 5248541480 1296457585 71969
				$\hat{h}$	1821668012 2563236651 6965526308 4134691342 7746539558 63
23	652	6	$p_1p_3p_5p_6$	h	4969630428 8186164902 1823347258 7791666535 5166346161 8759325372 3048193523 7025385660 9

row	q	s	primes of h		value of h (first line) and of $\hat{h} = L_q/h$ (second line)
				$\hat{h}$	3661148542 8978144899 9648484540 4339865824 9488531569 747383
24	652	6	$p_2p_6p_7$	h	9987854582 0619016616 3207525206 6156397151 4723064029 6982547272 5248541480 1296457585 71969
				$\hat{h}$	1821668012 2563236651 6965526308 4134691342 7746539558 63
25	652	6	$p_1p_2p_5p_6p_7$	h	1707564597 6599186234 6791963923 5332213109 2635154535 0157866559 1009624697 6487766358 3114059723 9849845918 7676528709 280887649
				$\hat{h}$	1065526611 885963703

The factorizations of L_q at the ten periods of Table A1 are as follows. The primes p_i are the known primitive prime divisors of F_{2q} that divide L_q, in increasing order; the other prime factors listed are not primitive. Every prime was proved prime as described in the proof of Theorem 11. The factorizations are complete except at $q = 604$, where a composite cofactor c remains.

$L_{188} = 7 \cdot p_1p_2p_3$, where p_1 = 18049; p_2 = 100769; p_3 = 153037630649666194962091443041.

$L_{236} = 7 \cdot p_1p_2p_3p_4$, where p_1 = 12743; p_2 = 13687; p_3 = 5974828049; p_4 = 2871307447 9853139217 0888873108 9.

$L_{244} = 7 \cdot p_1p_2p_3$, where p_1 = 487; p_2 = 52471477541626010209; p_3 = 55009022301464381514054890 47.

$L_{268} = 7 \cdot p_1p_2p_3$, where p_1 = 4289; p_2 = 6387083201; p_3 = 5320236363 4582214703 8743367122 4543829638 89.

$L_{364} = 7^2 \cdot 103 \cdot 14503 \cdot 102193207 \cdot p_1p_2p_3p_4p_5$, where p_1 = 727; p_2 = 193649; p_3 = 800801; p_4 = 8303168327; p_5 = 1683719541 5021202231 4165102991 8429841.

$L_{532} = 7^2 \cdot 14503 \cdot 1091346396980401 \cdot p_1p_2p_3p_4$, where p_1 = 1063; p_2 = 2129; p_3 = 996967; p_4 = 8677974263 7995804396 8754111726 5562037539 4209537853 5126252231 5612418716 97593889.

$L_{556} = 7 \cdot p_1p_2p_3p_4$, where p_1 = 41143; p_2 = 42243409662449; p_3 = 11659023981678323993980 8689; p_4 = 1109974042 8816594955 5811153489 8451109572 2291048840 3176792045 4249040312 87.

$L_{604} = 7 \cdot p_1p_2p_3p_4 \cdot c$, where p_1 = 3623; p_2 = 895127; p_3 = 1034748641; p_4 = 353413035677611688249089; c = 2038760687 5232156161 9096855107 4592740338 8538939493 5097154848 8874804560 3379014147 7169, a composite number of 84 digits that is not yet factored.

$L_{652} = 7 \cdot p_1p_2p_3p_4p_5p_6p_7$, where p_1 = 1303; p_2 = 2609; p_3 = 81456967; p_4 = 1868693287; p_5 = 1312080604 8664680673 8527589120 7; p_6 = 3568542934 8848301447 4970191912 5982472887; p_7 = 1072771445 5727240256 7111366834 8292251690 743.

$L_{692} = 7 \cdot p_1p_2p_3p_4$, where p_1 = 70423732801; p_2 = 116647040270104684001; p_3 = 3132686123916859420769; p_4 = 2311209770 2964602725 4151690471 9092597079 3451425095 8284540195 9357302863 6894200093 7390604570 09.

Appendix B. Notation

Table B1. Notation and terms, with the place of their definition.

Symbol or term	Meaning
R; g	cofactor map; step multiplier $\gcd(\beta, \alpha+1)$ of a state (α, β); a step with $g > 1$ is an amplification (Section 2)
q, G	least period of a cycle; cycle multiplier, the product of its step multipliers (Section 2)
F_n, L_n; φ, ψ	Fibonacci and Lucas numbers; $(1 \pm \sqrt{5})/2$
w', $N(w)$, $c(w)$	conjugate, norm and content of $w \in \mathbb{Z}[\varphi]$ (Section 2)
$\pi(r)$	order of φ modulo the prime r, the Pisano period (Section 2)
$(q_1, q_2; g_1, g_2)$	pattern: the lengths of segments 1 and 2 and the multipliers ending them (Definition 1)
t, D	$G - L_q$; $G^2 - L_qG + (-1)^q = N(G - \varphi^q)$ (Definition 1)
Y, X	golden coordinates of the states after the second and after the first amplification (Lemma 2)
s, ℓ	lengths of the short and of the long segment, $s + \ell = q$ (Section 4)
h, $\hat{h}$	long multiplier; short multiplier L_q/h, when $G = L_q$ (Section 4)
σ^*, σ°	anchor states, after the amplifications ending the short and the long segment (Theorem 7, Lemma 10)
λ, μ	length of a segment and the multiplier ending it (Proposition 8)
τ, K, N_b	$G - \varphi^q$; $F_{q+q_2} + tF_{q_2}$; numerator of the φ-coordinate of Y (Lemma 13)
local conditions modulo r	$\varphi^i Y \not\equiv 1$ and $\varphi^i X \not\equiv 1 \pmod r$ along both segments (Section 3)
local survivor	a pattern that satisfies the local conditions modulo 2, 3, 5 and 7 (Section 3)
admissible period	$q \equiv 4$ or $20 \pmod{24}$ (Section 4)
A1; S1–S4; R1–R3	assumption of coprime seeds; scope labels; results restated from the earlier paper (Section 2)