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GCD-AUGMENTED FIBONACCI SEQUENCES: CLOSED FORMS FOR GENERALIZED SEEDS

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Abstract

We study the integer sequence defined by the recurrence $C_p(n) = C_p(n-1) + C_p(n-2) + \gcd(C_p(n-1), C_p(n-2))$ with seeds $C_p(1) = 1$, $C_p(2) = p$, generalizing OEIS sequence A083658 (the case $p = 1$). We establish closed-form expressions for two families of seeds. For even p, the sequence is purely geometric with base 2: $C_p(n) = (p+2) \cdot 2^{n-3}$ for $n \geq 3$ (Theorem 1). For odd p with $\gcd(p, 3) = 1$, the odd-indexed and even-indexed subsequences are each geometric with base 3, yielding $C_p(2k+1) = (p+2) \cdot 3^{k-1}$ and $C_p(2k) = 5(p+2) \cdot 3^{k-3}$, with the consecutive ratio alternating between 5/3 and 9/5 (Theorems 2–3). We prove a scaling property $C_{ka,kb}(n) = k \cdot C_{a,b}(n)$ reducing the study of arbitrary seed pairs to coprime seeds (Theorem 4), and characterize an anomalous regime for odd seeds divisible by 3 where the geometric structure breaks down (Proposition 1). All results are verified computationally for seeds $p = 1$ through 49.

Keywords: GCD-Fibonacci sequence; integer sequences; closed-form expressions; OEIS A083658; Fibonacci generalizations; greatest common divisor

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1. Introduction

The Fibonacci sequence, defined by $F(1) = F(2) = 1$ and $F(n) = F(n-1) + F(n-2)$ for $n \geq 3$, is among the most studied objects in combinatorial number theory. Its generalizations — including Lucas sequences, Tribonacci numbers, Pell numbers, Fibonacci polynomials, and Horadam's generalized sequences (1961) — form a rich family of linear recurrences whose asymptotic, algebraic, and combinatorial properties have been extensively investigated over the past century (Koshy, 2001).

A less-explored direction is the introduction of *nonlinear* correction terms into Fibonacci-type recurrences. In particular, the recurrence $C(n) = C(n-1) + C(n-2) + \gcd(C(n-1), C(n-2))$

augments the classical addition rule with a greatest-common-divisor term that depends on the arithmetic relationship between consecutive entries. This recurrence was first studied implicitly by Hanna (2003), who introduced the sequence $C(1) = C(2) = 1$, catalogued as OEIS A083658. The connection to the GCD-based formulation was made explicit by (Dale, 2025).

For seeds (1, 1), the sequence A083658 admits the closed form $C(2n) = 3^n$ and $C(2n+1) = 5 \cdot 3^{n-1}$, a remarkably clean result in which the nonlinear GCD term produces exact geometric growth with rational ratios — in sharp contrast to the irrational golden ratio $\varphi = (1+\sqrt{5})/2$ governing the classical Fibonacci sequence.

Despite the elegance of this result, the existing literature contains no investigation of the recurrence with seeds other than (1, 1). The works of Guyer and Mbirika (2021) study the GCD *of* sums of Fibonacci numbers — a related but structurally distinct problem — while the broader literature on Fibonacci generalizations (Horadam 1961, Koshy 2001) focuses on linear recurrences and does not incorporate GCD as an additive term. The GCD-Fibonacci recurrence with general seeds thus represents an unexplored family of nonlinear integer sequences.

Contributions. In this paper, we provide the first systematic study of the GCD-Fibonacci recurrence with seeds $(1, p)$ for arbitrary positive integers p. Our main results are:

- *Even seeds* (**Theorem 1**): For even p, the sequence is purely geometric with base 2 and closed form $C_p(n) = (p+2) \cdot 2^{n-3}$ for $n \geq 3$.
- *Odd coprime seeds* (**Theorems 2–3**): For odd p with $\gcd(p, 3) = 1$, the odd-indexed and even-indexed subsequences are each geometric with base 3, yielding $C_p(2k+1) = (p+2) \cdot 3^{k-1}$ and $C_p(2k) = 5(p+2) \cdot 3^{k-3}$, with the ratio alternating between 5/3 and 9/5.
- *Scaling* (**Theorem 4**): The recurrence satisfies $C_{ka,kb}(n) = k \cdot C_{a,b}(n)$, so the study of arbitrary seed pairs reduces to coprime seeds.
- *Anomalous regime* (**Proposition 1**): For odd p with $3 \mid p$, the base-3 structure breaks down, and the sequence exhibits extended non-geometric transients whose characterization remains an open problem.

These results reveal a striking structural dichotomy: even seeds produce base-2 geometry driven by the factor $\gcd(p+2, p) = 2$, while odd coprime seeds produce base-3 geometry driven by the identity $5 + 3 + 1 = 9 = 3^2$. The GCD correction term acts as an arithmetic "resonance detector" — when consecutive terms share a large common factor, growth accelerates; when they are coprime, the correction is minimal.

Organization. Section 2 establishes notation, definitions, and preliminary lemmas. Section 3 presents the main closed-form results for even and odd coprime seeds, including the asymptotic ratio analysis. Section 4 develops extensions: the scaling theorem, recovery of A083658 as a special case, and the anomalous regime. Section 5 discusses the results and identifies open problems.

2. Preliminaries

Throughout this paper, gcd(a, b) denotes the greatest common divisor of positive integers a and b, and we adopt the convention that all sequence indices begin at $n = 1$.

Definition 1 (GCD-Fibonacci sequence). Let p be a positive integer. The *GCD-Fibonacci sequence with parameter p*, denoted $\{C_p(n)\}_n \geq 1$, is defined by the recurrence

$$C_p(1) = 1,\ C_p(2) = p,\ C_p(n) = C_p(n-1) + C_p(n-2) + \gcd(C_p(n-1), C_p(n-2)) \quad \text{for } n \geq 3.$$

When the parameter p is clear from context, we write $C(n)$ in place of $C_p(n)$.

The recurrence augments the classical Fibonacci addition rule with a GCD correction term, creating a sequence whose growth rate depends on the arithmetic relationship between consecutive terms rather than on a fixed linear recurrence. The special case $p = 1$ recovers OEIS sequence A083658 (Dale, 2025), for which Hanna (2003) established the closed form $C_1(2n) = 3^n$ and $C_1(2n+1) = 5 \cdot 3^{n-1}$.

Definition 2 (General seeds). For positive integers a and b, define the *GCD-Fibonacci sequence with seeds* (a, b), denoted $\{C_{a,h}(n)\}_n \geq 1$, by

$$C_{a,h}(1) = a, \quad C_{a,h}(2) = b, \quad C_{a,h}(n) = C_{a,h}(n-1) + C_{a,h}(n-2) + \gcd(C_{a,h}(n-1), C_{a,h}(n-2)).$$

Thus $C_p(n) = C_{1,p}(n)$, and the single-parameter family is embedded in the two-parameter family.

We collect several elementary properties of the greatest common divisor that will be used repeatedly in the proofs of our main results.

Lemma 1. *For any positive integer p:*

(i) gcd(p + 2, p) = gcd(2, p).

(ii) If p is even, then gcd(p + 2, p) = 2.

(iii) If p is odd, then gcd(p + 2, p) = 1.

Proof. Part (i) follows from the Euclidean algorithm: $\gcd(p + 2, p) = \gcd(p + 2 - p, p) = \gcd(2, p)$. Part (ii): if p is even then $2 \mid p$, so $\gcd(2, p) = 2$. Part (iii): if p is odd then $2 \nmid p$, so $\gcd(2, p) = 1$. □

Lemma 2. *For positive integers m and k, gcd(km, m) = m.*

Proof. Since $m \mid km$, we have $\gcd(km, m) = m \cdot \gcd(k, 1) = m$. □

Lemma 3 (Homogeneity of GCD). *For any positive integers a, b, and k: gcd(ka, kb) = k · gcd(a, b).*

Proof. This is a standard property; see, e.g., (Koshy, 2001)'s Theorem 2.4. □

Lemma 1 governs the initial step of the recurrence and is responsible for the fundamental even/odd dichotomy observed in our main results. Lemma 2 ensures that once consecutive terms achieve a constant ratio, the GCD correction term equals the smaller term, locking the sequence into geometric growth. Lemma 3 underpins the scaling theorem (Theorem 4) by guaranteeing that the GCD correction respects uniform scaling of the seeds.

Table 1. Notation used throughout the paper

Symbol	Definition
$C_p(n)$	GCD-Fibonacci sequence with seeds (1, p)
$C_{a,b}(n)$	GCD-Fibonacci sequence with seeds (a, b)
p	Seed parameter (positive integer)
gcd(a, b)	Greatest common divisor of a and b

Remark 1. The GCD correction term $\gcd(C(n-1), C(n-2))$ can be interpreted as a measure of arithmetic *resonance* between consecutive terms. When consecutive terms share a large common factor, the correction significantly accelerates growth; when they are coprime, the recurrence reduces to $C(n) = C(n-1) + C(n-2) + 1$, a shifted Fibonacci recurrence. This resonance mechanism is ultimately responsible for the sharp behavioral dichotomy between even and odd seeds established in Sections 3 and 4.

3. Main Results

We now present the closed-form expressions for the GCD-Fibonacci sequence $C_p(n)$. The results divide naturally into two regimes determined by the parity of the seed parameter p.

3.1 Even Seeds: Geometric Base 2

The even-seed case yields a single geometric progression, making it the simpler of the two regimes.

Theorem 1 (Even-seed closed form). *Let p be an even positive integer. Then:*

$C_p(n) = (p+2) \cdot 2^{n-3}$ for all $n \geq 3$.

Consequently, $C_p(n+1)/C_p(n) = 2$ *for all* $n \geq 3$*, and the sequence is purely geometric with base 2 after the initial two terms.*

Proof. We proceed by strong induction on n.

Base cases. For $n = 3$: $C_p(3) = p + 1 + \gcd(p, 1) = p + 2$. Since $(p+2) \cdot 2^0 = p+2$, the formula holds. For $n = 4$: $C_p(4) = (p+2) + p + \gcd(p+2, p) = 2p + 2 + 2 = 2(p+2)$, where we used Lemma 1(ii). Since $(p+2) \cdot 2^1 = 2(p+2)$, the formula holds.

Inductive step. Suppose the formula holds for all indices from 3 to $n-1$, where $n \geq 5$. Then $C_p(n-1) = (p+2) \cdot 2^{n-4}$ and $C_p(n-2) = (p+2) \cdot 2^{n-5}$, so that $C_p(n-1) = 2 \cdot C_p(n-2)$. By Lemma 2, $\gcd(C_p(n-1), C_p(n-2)) = C_p(n-2)$. Therefore:

$C_p(n) = 2 \cdot C_p(n-2) + 2 \cdot C_p(n-2) = 4 \cdot C_p(n-2) = (p+2) \cdot 2^{n-3}$. □

Corollary 1. *For even p, the sequence $\{C_p(n)\}_{n\geq 3}$ is a geometric progression with common ratio 2 and initial value p+2.*

The even-seed regime admits a clean structural explanation. After the transient terms $C_p(1) = 1$ and $C_p(2) = p$, every consecutive pair satisfies $C_p(n-1) = 2 \cdot C_p(n-2)$, so the GCD correction always equals the smaller term. The recurrence effectively becomes $C_p(n) = 2C_p(n-1)$, yielding the factor of 2 per step.

3.2 Odd Coprime Seeds: Dual-Regime Base 3

When p is odd with $\gcd(p, 3) = 1$, the sequence exhibits a richer structure: the odd-indexed and even-indexed subsequences each form geometric progressions with base 3, but with different multiplicative constants.

We first compute the initial terms explicitly to establish the base cases and reveal the pattern.

Table 2. First terms of $C_p(n)$ for odd p with $\gcd(p, 3) = 1$

n	$C_p(n)$	gcd used
1	1	—
2	p	—
3	p + 2	gcd(p, 1) = 1
4	2p + 3	gcd(p+2, p) = 1
5	3(p+2)	gcd(2p+3, p+2) = 1
6	5(p+2)	gcd(3(p+2), 2p+3) = 1

The gcd computations for $n = 5$ and $n = 6$ require verification. For $n = 5$: $\gcd(2p+3, p+2) = \gcd(2p+3 - 2(p+2), p+2) = \gcd(-1, p+2) = 1$. For $n = 6$: we have $2(3p+6) - 3(2p+3) = 3$, so $\gcd(3p+6, 2p+3) = \gcd(2p+3, 3)$. Since $\gcd(p, 3) = 1$ and p is odd, one checks that $2p + 3 \not\equiv 0 \pmod 3$ in both residue classes, giving $\gcd(2p+3, 3) = 1$. See Figure 1.

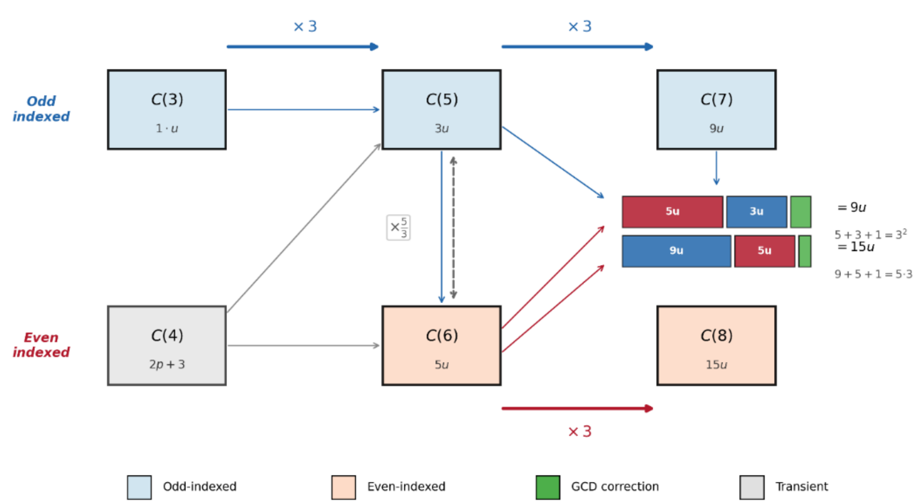

Figure 1. Term-building structure for the odd coprime regime with unit $u = p + 2$. Odd-indexed and even-indexed subsequences each grow by factor 3 every two steps, driven by the additive identities 5 + 3 + 1 =

$9 = 3^2$ and $9 + 5 + 1 = 15 = 5 \cdot 3$. Decomposition bars show the three contributions (previous even term, previous odd term, GCD correction) to each new term.

Theorem 2 (Odd-indexed closed form). *Let p be odd with gcd(p, 3) = 1. Then:*

$C_p(2k+1) = (p+2) \cdot 3^{k-1}$ for all $k \geq 1$.

Theorem 3 (Even-indexed closed form). *Let p be odd with gcd(p, 3) = 1. Then:*

$C_p(2k) = 5(p+2) \cdot 3^{k-3}$ for all $k \geq 3$.

Additionally, $C_p(4) = 2p + 3$.

Theorems 2 and 3 are proved simultaneously by mutual strong induction.

Proof of Theorems 2 and 3. The base cases $C_p(3) = p + 2 = (p+2) \cdot 3^0$, $C_p(5) = 3(p+2) = (p+2) \cdot 3^1$, and $C_p(6) = 5(p+2) = 5(p+2) \cdot 3^0$ are verified in Table 2.

Inductive step for Theorem 2. Assume Theorem 2 holds for all odd-indexed terms up to $C_p(2k-1)$ and Theorem 3 holds for all even-indexed terms up to $C_p(2k)$, where $k \geq 3$. Then $C_p(2k) = 5(p+2) \cdot 3^{k-3}$ and $C_p(2k-1) = (p+2) \cdot 3^{k-2}$. Factoring out $(p+2) \cdot 3^{k-3}$:

$\gcd(C_p(2k), C_p(2k-1)) = (p+2) \cdot 3^{k-3} \cdot \gcd(5, 3) = (p+2) \cdot 3^{k-3}$. Therefore:

$C_p(2k+1) = (p+2) \cdot 3^{k-3}(5 + 3 + 1) = (p+2) \cdot 3^{k-3} \cdot 9 = (p+2) \cdot 3^{k-1}$.

Inductive step for Theorem 3. Assume Theorem 2 holds up to $C_p(2k-1)$ and Theorem 3 holds up to $C_p(2k-2)$, To prove the result for $C_p(2k)$, we let $k \geq 4$, utilizing $C_p(5)$ and $C_p(6)$ as the verified base cases.

Then $C_p(2k-1) = (p+2) \cdot 3^{k-2}$ and $C_p(2k-2) = 5(p+2) \cdot 3^{k-4}$. Factoring out $(p+2) \cdot 3^{k-4}$:

$\gcd(C_p(2k-1), C_p(2k-2)) = (p+2) \cdot 3^{k-4} \cdot \gcd(9, 5) = (p+2) \cdot 3^{k-4}$. Therefore:

$C_p(2k) = (p+2) \cdot 3^{k-4}(9 + 5 + 1) = (p+2) \cdot 3^{k-4} \cdot 15 = 5(p+2) \cdot 3^{k-3}$. □

Remark 2. The mutual induction succeeds because the identity $5 + 3 + 1 = 9 = 3^2$ governs the odd-indexed step and $9 + 5 + 1 = 15 = 5 \cdot 3$ governs the even-indexed step — both producing the correct powers of 3 and the factor of 5.

3.3 Asymptotic Ratio and Comparison with Fibonacci

Corollary 2 (Alternating ratio). *Let p be odd with gcd(p, 3) = 1. For* $n \geq 5$*:*
$C_p(n+1)/C_p(n) = 5/3$ if n is odd; $9/5$ if n is even.

Consequently, $C_p(n+2)/C_p(n) = 3$ *for all* $n \geq 5$.

Proof. For odd $n = 2k-1$ with $k \geq 3$: $C_p(2k)/C_p(2k-1) = 5(p+2) \cdot 3^{k-3} / [(p+2) \cdot 3^{k-2}] = 5/3$. For even $n = 2k$ with $k \geq 3$: $C_p(2k+1)/C_p(2k) = (p+2) \cdot 3^{k-1} / [5(p+2) \cdot 3^{k-3}] = 9/5$. The two-step ratio is $(5/3) \cdot (9/5) = 3$. □

Table 3. First 12 terms of $C_p(n)$ for selected seed values

n	p=1	p=2	p=5	p=7	p=10
1	1	1	1	1	1
2	1	2	5	7	10
3	3	4	7	9	12
4	5	8	13	17	24
5	9	16	21	27	48
6	15	32	35	45	96
7	27	64	63	81	192
8	45	128	105	135	384
9	81	256	189	243	768
10	135	512	315	405	1536
11	243	1024	567	729	3072
12	405	2048	945	1215	6144

The even seed $p = 10$ displays exact doubling from $n = 3$ onward. The odd coprime seeds $p = 1$, 5, 7 display the alternating ratio pattern from $n = 5$ onward. See Figure 2.

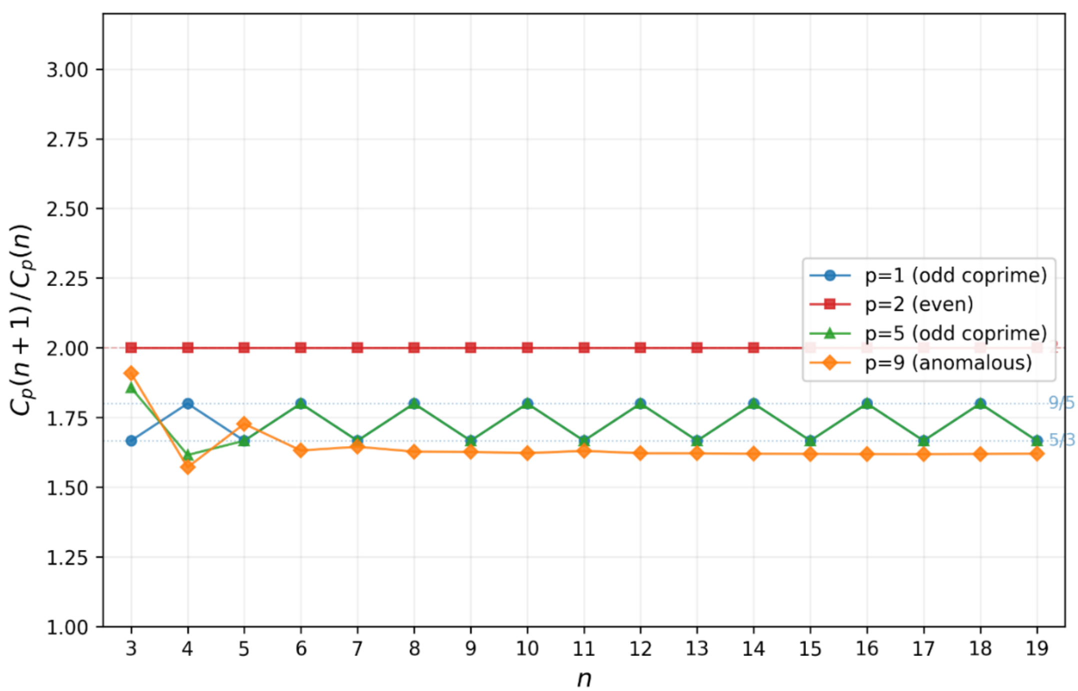

Figure 2. Plot of $C_p(n+1)/C_p(n)$ for $n = 3, \ldots, 20$ for seeds $p = 1$ (odd coprime), $p = 2$ (even), $p = 5$ (odd coprime), $p = 9$ (anomalous). Shows the three behavioral regimes: constant ratio 2 (even), alternating 5/3 and 9/5 (odd coprime), and irregular (anomalous)

Remark 3. The contrast with the Fibonacci sequence is striking. The classical ratio $F(n+1)/F(n) \to \varphi = (1+\sqrt{5})/2 \approx 1.618$ is irrational and monotonically approached. Here, the ratios 5/3 and 9/5 are rational and exactly achieved after a finite transient. Their geometric mean $\sqrt{[(5/3)(9/5)]} = \sqrt{3}$

≈ 1.732 is irrational — an inversion of the Fibonacci paradigm, where an irrational limit is approached through rational approximants.

4. Extensions

The closed forms established in Section 3 apply to seeds of the form $(1, p)$. In this section, we extend the theory in three directions: a scaling property that relates arbitrary seed pairs to the single-parameter family, recovery of the original OEIS sequence as a special case, and a characterization of the anomalous regime arising when $3 \mid p$.

4.1 Scaling Property

Theorem 4 (Scaling). *For any positive integers a, b, and k:*

$$C_{ka,kb}(n) = k \cdot C_{a,b}(n) \quad \text{for all } n \geq 1.$$

Proof. By induction on n. The base cases are immediate: $C_{ka,kb}(1) = ka = k \cdot C_{a,b}(1)$ and $C_{ka,kb}(2) = kb = k \cdot C_{a,b}(2)$.

For the inductive step, assume $C_{ka,kb}(j) = k \cdot C_{a,b}(j)$ for all $j \leq n-1$. Then:

$$C_{ka,kb}(n) = k \cdot C_{a,b}(n-1) + k \cdot C_{a,b}(n-2) + \gcd(k \cdot C_{a,b}(n-1), k \cdot C_{a,b}(n-2)).$$

By Lemma 3 (homogeneity of GCD):

$$= k \cdot (C_{a,b}(n-1) + C_{a,b}(n-2) + \gcd(C_{a,b}(n-1), C_{a,b}(n-2))) = k \cdot C_{a,b}(n). \quad \square$$

Corollary 3 (Reduction to coprime seeds). *For any positive integers a and b with d = gcd(a, b):*

$$C_{a,b}(n) = d \cdot C_{a/d,\, b/d}(n).$$

In particular, the study of all two-seed GCD-Fibonacci sequences reduces to the study of sequences with coprime seeds.

Theorem 4 reveals that the GCD-Fibonacci recurrence is homogeneous of degree one in the seeds: scaling both seeds by k scales the entire sequence by k, preserving all ratios and structural properties. This is a direct consequence of the homogeneity of the GCD function itself (Lemma 3).

4.2 Recovery of A083658

Corollary 4 (Recovery of A083658). *Setting p = 1 in Theorems 2 and 3:*

$$C_1(2k+1) = 3^k \text{ for } k \geq 1, \qquad C_1(2k) = 5 \cdot 3^{k-2} \text{ for } k \geq 3.$$

This recovers the closed form for OEIS A083658 established by Hanna (2003).

Proof. Since $p = 1$ is odd and $\gcd(1, 3) = 1$, Theorems 2 and 3 apply. Substituting $p = 1$: $C_1(2k+1) = (1+2) \cdot 3^{k-1} = 3^k$ and $C_1(2k) = 5(1+2) \cdot 3^{k-3} = 5 \cdot 3^{k-2}$. □

The recovery of A083658 as a special case confirms that the original sequence's closed form is not an isolated phenomenon but part of a broader family of exact solutions parametrized by the seed p.

4.3 Anomalous Regime: Seeds Divisible by 3

The closed forms of Theorems 2 and 3 require $\gcd(p, 3) = 1$. When p is odd with $3 \mid p$, the sequence enters a qualitatively different regime.

Proposition 1 (Anomalous regime). *Let p be odd with $3 \mid p$. Then $C_p(3) = p + 2 \equiv 2 \pmod 3$, and the gcd stabilization mechanism breaks down at $n = 6$: the condition $\gcd(2p+3, 3) = 1$ fails, since $3 \mid p$ implies $2p + 3 \equiv 0 \pmod 3$.*

Table 4. First 10 terms for selected anomalous seeds (p odd, $3 \mid p$)

n	p=3	p=9	p=15	p=21	p=27
1	1	1	1	1	1
2	3	9	15	21	27
3	5	11	17	23	29
4	9	21	33	45	57
5	15	33	51	69	87
6	27	57	87	117	147
7	45	93	141	189	243
8	81	153	231	315	399
9	135	249	375	513	651
10	243	405	609	837	1053

Computational investigation reveals that some seeds ($p = 3$, $p = 21$) eventually enter a geometric regime, while others ($p = 9$, $p = 15$, $p = 27$) exhibit extended non-geometric transients through at least $n = 30$.

Open Question. *Characterize which odd multiples of 3 eventually enter a geometric regime, and determine the transient length as a function of p.*

This anomalous behavior arises because the condition $\gcd(p, 3) = 1$ is essential to the proof mechanism: at $n = 6$, the gcd computation $\gcd(3(p+2), 2p+3)$ reduces to $\gcd(2p+3, 3)$, which equals 3 rather than 1 when $3 \mid p$. The resulting larger GCD correction disrupts the delicate balance $5 + 3 + 1 = 9$ that drives the base-3 geometry. The anomalous regime therefore represents a genuine structural boundary of the theory, not merely a gap in the proof technique.

5. Discussion and Conclusions

5.1 Summary of Contributions

We have provided the first systematic study of the GCD-Fibonacci recurrence $C_p(n) = C_p(n-1) + C_p(n-2) + \gcd(C_p(n-1), C_p(n-2))$ with seeds $(1, p)$ for arbitrary positive integers p. The principal results are:

- **Theorem 1:** For even p, the sequence is purely geometric with base 2 and closed form $C_p(n) = (p+2) \cdot 2^{n-3}$ for $n \geq 3$.
- **Theorems 2–3:** For odd p with $\gcd(p, 3) = 1$, dual-regime closed form with ratios alternating between 5/3 and 9/5.
- **Theorem 4:** Scaling property $C_{ka,kb}(n) = k \cdot C_{a,b}(n)$ reduces arbitrary seeds to coprime seeds.
- **Proposition 1:** Anomalous regime for odd p with $3 \mid p$.

All results have been verified computationally for seeds $p = 1$ through 49, with terms computed up to $n = 23$.

5.2 The Structural Dichotomy

The most striking feature of the GCD-Fibonacci recurrence is the sharp behavioral dichotomy governed by the parity and divisibility properties of the seed p:

- **Even seeds** lock into geometric growth with base 2, because the initial GCD value $\gcd(p+2, p) = 2$ establishes a doubling ratio that perpetuates itself through Lemma 2.
- **Odd coprime seeds** lock into geometric growth with base 3, because the initial coprimality conditions force the GCD correction to stabilize at a predictable fraction of the smaller term, and the arithmetic identity $5 + 3 + 1 = 9 = 3^2$ drives the inductive machinery.
- **Odd seeds divisible by 3** disrupt this stabilization at $n = 6$, where $\gcd(2p+3, 3) = 3$ rather than 1, injecting an anomalous correction that prevents geometric lock-in.

This trichotomy is entirely determined by two arithmetic properties of the seed: parity (even vs. odd) and residue modulo 3 (coprime to 3 vs. divisible by 3). See Figure 3.

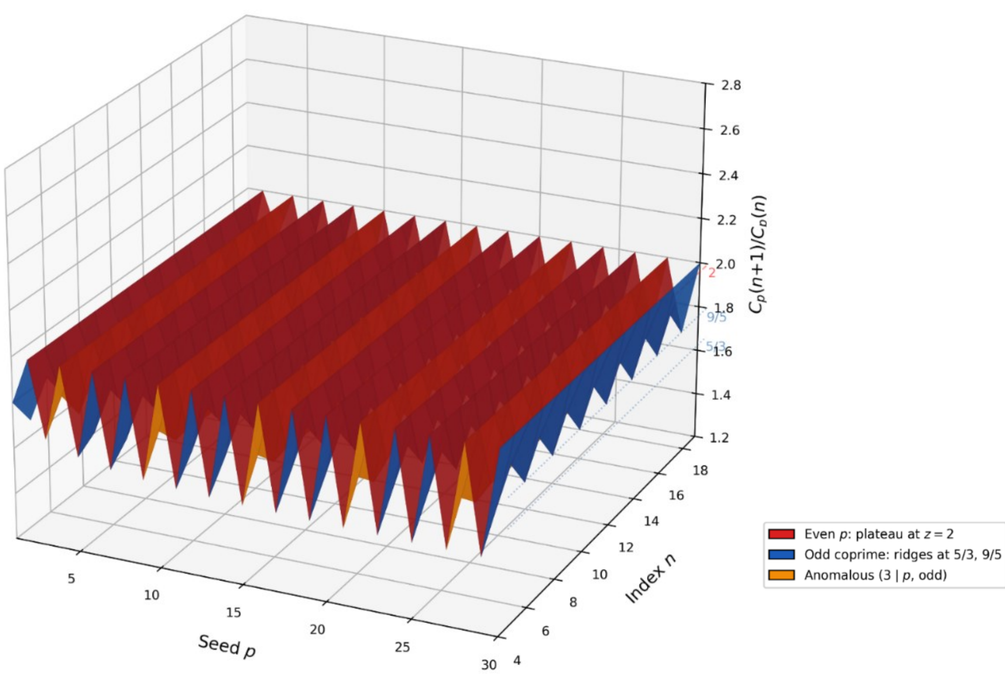

Figure 3. Ratio landscape $C_p(n+1)/C_p(n)$over the (p, n)plane for $p = 1, \dots, 30$and $n = 4, \dots, 19$. Even seeds (red) form a flat plateau at $z = 2$. Odd coprime seeds (blue) oscillate between $5/3$and $9/5$, forming

sawtooth ridges. Anomalous seeds with $3 \mid p$(gold) exhibit irregular terrain reflecting extended non-geometric transients

5.3 Comparison with Classical Fibonacci

The GCD-Fibonacci recurrence inverts several features of the classical Fibonacci sequence. The classical ratio $F(n+1)/F(n)$ converges to the irrational golden ratio $\varphi \approx 1.618$ through rational approximants. In contrast, the GCD-Fibonacci ratios 5/3 and 9/5 are rational and exactly achieved after a finite transient, while their geometric mean $\sqrt{3}$ is irrational.

Fibonacci growth is governed by the dominant eigenvalue of a 2×2 linear system; GCD-Fibonacci growth is governed by the nonlinear GCD correction, which selects different geometric bases depending on seed arithmetic. The Fibonacci sequence is universal — all seeds (a, b) with $a, b > 0$ produce the same asymptotic ratio φ — while the GCD-Fibonacci asymptotic behavior is seed-dependent.

5.4 The GCD as Resonance Detector

The GCD correction term $\gcd(C(n-1), C(n-2))$ can be interpreted as a measure of arithmetic resonance between consecutive terms. When consecutive terms share a large common factor — as in the even-seed regime where each term is twice its predecessor — the correction significantly accelerates growth. When they are coprime, the correction reduces to 1 and the recurrence approximates a shifted Fibonacci rule. The transition from coprime to non-coprime consecutive terms marks the onset of geometric lock-in.

5.5 Open Problems

1. **Anomalous regime characterization.** For odd p with $3 \mid p$, computational evidence shows that some seeds ($p = 3$, $p = 21$) eventually enter a geometric regime while others ($p = 9$, $p = 15$, $p = 27$) exhibit extended transients through at least $n = 30$. A complete characterization remains open.
2. **General coprime seeds (a, b) with $a > 1$.** The scaling theorem reduces arbitrary seeds to coprime seeds, but our closed forms cover only seeds of the form $(1, p)$. The behavior of $C_{a,b}(n)$ with $\gcd(a, b) = 1$ and $a > 1$ is unexplored.
3. **Higher-order generalizations.** Recurrences involving sums of $r \geq 3$ preceding terms plus their GCD may exhibit analogous geometric lock-in phenomena.
4. **Algebraic structure.** The identities $5 + 3 + 1 = 9$ and $9 + 5 + 1 = 15$ that drive the odd coprime regime suggest deeper algebraic structure. Whether these arise from a more general pattern governing GCD-augmented linear recurrences is unknown.

5.6 Limitations

Our computational verification extends to $p = 49$ and $n = 23$. While the proofs are complete and do not depend on computational evidence, independent verification for larger parameter ranges would strengthen confidence. The anomalous regime (Proposition 1) is characterized only computationally; a rigorous proof of eventual stabilization or non-stabilization for specific seeds remains an open challenge.

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7. References

1. Dale, H. P. (2025, September). *GCD formulation of A083658*. OEIS. https://oeis.org/A083658
2. Guyer, D., & Mbirika, A. (2021). GCD of sums of consecutive Fibonacci, Lucas, and generalized Fibonacci numbers. *Journal of Integer Sequences*, *24*(9), Article 21.9.8. https://cs.uwaterloo.ca/journals/JIS/VOL24/Mbirika/mbirika3.html
3. Hanna, P. D. (2003). *Sequence A083658*. The On-Line Encyclopedia of Integer Sequences. https://oeis.org/A083658
4. Horadam, A. F. (1961). A generalized Fibonacci sequence. *American Mathematical Monthly*, *68*(5), 455–459. https://doi.org/10.2307/2311088
5. Koshy, T. (2001). *Fibonacci and Lucas numbers with applications*. Wiley-Interscience.