

Exactly Three Single-Amplification Cycles in GCD-Augmented Fibonacci Recurrences

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Abstract

The integer recurrence $C(n) = C(n-1) + C(n-2) + \gcd(C(n-1), C(n-2))$ reduces, after division by the gcd of consecutive terms, to a map on coprime cofactor pairs. A step of this map is an amplification when the gcd grows. We determine all cycles of the map with exactly one amplification per period. There are exactly three: the fixed point (1, 2), the 2-cycle $(3, 5) \rightarrow (5, 9)$, and a new 4-cycle $(29, 47) \rightarrow (47, 77) \rightarrow (77, 125) \rightarrow (125, 203)$. Their cycle multipliers are 2, 3 and 7. A sequence whose cofactor pairs reach one of these cycles therefore eventually grows by exactly 2 per step, 3 per two steps or 7 per four steps. The proof shifts the cofactors by one, which makes the dynamics between amplifications linear in the Fibonacci matrix. The unique solution of the resulting linear system has an explicit Fibonacci norm, which forces the cycle multiplier to be a Lucas number or one more than a Lucas number; gcd identities for Fibonacci and Lucas numbers then exclude every period except 1, 2 and 4. As consequences, we give an algebraic proof that the period-2 orbit is unique, and we correct a claim in the author's earlier work that no other periodic orbits exist. We also show that the seeds (1, 9), (1, 15) and (1, 27), described as non-geometric in the author's earlier work, grow by exactly 7 every four steps. For cycles with any number of amplifications, the same linearization gives a cycle equation and shows that the cycle multiplier exceeds φ^q , where q is the period and φ the golden ratio. In a census of 8,771 coprime seeds, the new cycle attracts 35.9%.

Keywords: GCD-augmented Fibonacci recurrence; periodic orbits; cycle equation; Lucas numbers; Fibonacci matrix; Leonardo numbers; Collatz-type maps; integer sequences; OEIS A083658

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1. Introduction

The recurrence $C(n) = C(n-1) + C(n-2) + \gcd(C(n-1), C(n-2))$ adds to the Fibonacci rule the greatest common divisor of the two preceding terms. With seeds $(1,1)$ it is OEIS A083658 [1], which grows by exactly 3 every two steps. For seeds $(1, p)$, the author's earlier work [2] proved that the growth is exactly geometric: ratio 2 for even p , and alternating ratios $5/3$ and $9/5$ for odd p prime to 3. It also isolated an anomalous class, odd p divisible by 3, for which no closed form was found.

The author's sequel [3] explained these regimes by dividing consecutive terms by their gcd. The resulting coprime cofactor pairs follow a map whose periodic orbits $(1,2)$ and $(3,5) \leftrightarrow (5,9)$ produce the ratios 2 and 3. It concluded that no other coprime periodic orbits exist [3, Corollary 5], while listing higher-period orbits as an open question.

We classify the single-amplification cycles of this cofactor map, that is, the cycles in which the gcd grows exactly once per period. Our main result (Theorem 10) is that there are exactly three: the two above and a 4-cycle $(29,47) \rightarrow (47,77) \rightarrow (77,125) \rightarrow (125,203)$, along which the sequence grows by exactly 7 every four steps. To the author's knowledge, the 4-cycle and the resulting regime $C(n+4) = 7C(n)$ had not been reported when this work was submitted: a search of the OEIS, arXiv and the journal literature on 7 October 2026 found no earlier report of the period-4 cycle, nor of a classification of single-amplification cycles for this recurrence. The 4-cycle corrects [3, Corollary 5]; the correction notices issued for [2] and [3] on 9 October 2026 record this and the related corrections.

The new cycle is not rare. It is reached by the anomalous seeds $p = 9, 15, 27$ of [2] (Proposition 12), by 35.9% of the coprime seeds with entries at most 120, and by 38.4% of those with entries at most 200 (Table 2). Along the way we obtain a purely algebraic proof of the uniqueness of the period-2 orbit [3, Theorem 8]. We also show that Corollary 5 of [3] is correct for periods at most 2 (Corollary 11).

The method is linear algebra over the Fibonacci matrix. Shifting the cofactors by one makes the dynamics between amplifications linear (Lemma 1), so a single-amplification cycle solves a 2×2 linear system (Lemma 3). The solution has an explicit norm (Lemma 5). Positivity and integrality then force the cycle multiplier of a cycle of period q to be the Lucas number L_q or $L_q + 1$ (Theorem 7). Finally, gcd identities for Fibonacci and Lucas numbers exclude every period except 1, 2 and 4 (Lemmas 8 and 9). The linearization does not depend on the number of amplifications: every cycle solves a linear system of the same type, and its cycle multiplier exceeds φ^q , where φ is the golden ratio (Proposition 13).

This places the problem in the framework of Greene and Niedzielski [4], who studied periodic integer solutions of $a_n = r(a_{n-1} + a_{n-2})$ when this value is an integer and $a_n = a_{n-1} + a_{n-2}$ otherwise, for a fixed rational r . There each step is one of two fixed 2×2 matrices, and periodic solutions correspond to products of these matrices with eigenvalue 1, found by computer search and in infinite families built from Fibonacci and Lucas identities. Here a single matrix suffices, the divisor is a gcd read off the state, and the dynamics is affine; the divisor becomes an unknown, but for one amplification the linear system can be solved in closed form and its

integrality settled by a norm form. Related periodic-or-divergent behaviour occurs in other nonlinear integer recurrences [5, 6] and in Collatz-type maps [7].

The closest methodological precedent is the classification of cycles of the $3x + 1$ map by the number of “events” per period. Böhm and Sontacchi [8] observed that, for a map that is affine on each of two classes, a cycle with a prescribed sequence of classes solves one linear equation, so each pattern carries at most one cycle. Steiner [9] proved that the trivial cycle $\{1, 2\}$ is the only 1-cycle, or circuit: a cycle formed by one increasing run of odd numbers followed by one decreasing run of even numbers. Simons and de Weger [10] proved that there are no nontrivial m -cycles, formed by m such runs, for $m \leq 68$. A single-amplification cycle, q Fibonacci-type steps with a division by g after the last one, is the analogue of a circuit, and cycles with several amplifications are the analogue of m -cycles. The decisive step differs. For the $3x + 1$ map the divisor is always 2, and circuits, and m -cycles with $m \leq 68$, are excluded by transcendence methods [9, 10]. Here the divisor is part of the unknown, the cycle equation is a 2×2 system whose determinant is a norm from $\mathbb{Q}(\sqrt{5})$, and the Fibonacci norm form, with gcd identities for Fibonacci and Lucas numbers, settles the single-amplification case by elementary means.

The recurrence is therefore a useful model case beyond the questions left open in [2] and [3]. Its cycles with one event per period can be classified completely and by elementary means, which is rare for maps of Collatz type. The method uses only affine dynamics between divisions and a divisor read off the state, so it applies to other conditional-division maps; Section 6 carries it over to a Tribonacci analogue. And the classification gives exact growth laws $C(n + q) = G C(n)$, obeyed by 83.1% of the coprime seeds with entries at most 120 and by 78.1% of those with entries at most 200 (Table 2).

Section 2 fixes notation. Section 3 linearizes the dynamics, Section 4 solves the cycle equation, and Section 5 proves the classification and its consequences. Section 6 extends the cycle equation to several amplifications and discusses limitations.

2. Preliminaries

Let a, b be positive integers and put $C(1) = a$, $C(2) = b$ and $C(n) = C(n - 1) + C(n - 2) + \gcd(C(n - 1), C(n - 2))$ for $n \geq 3$. Since $C_{ka, kb} = k C_{a, b}$ [2, Theorem 4], it suffices to consider the following.

A1. The seeds a, b are positive integers with $\gcd(a, b) = 1$.

For $n \geq 2$ let $d_n = \gcd(C(n - 1), C(n))$ and call the coprime pair $\sigma_n = (\alpha_n, \beta_n) = (C(n - 1)/d_n, C(n)/d_n)$ the *state* at index n . For a coprime pair (α, β) of positive integers define the step multiplier $g = g(\alpha, \beta) = \gcd(\beta, \alpha + 1)$ and the map

$$R(\alpha, \beta) = \left(\frac{\beta}{g}, \frac{\alpha + \beta + 1}{g} \right).$$

The cofactor factorization of [3, Theorem 5 and Lemma 4] reads, in this notation, $C(n+1) = d_n(\alpha_n + \beta_n + 1)$, $\sigma_{n+1} = R(\sigma_n)$ and $d_{n+1} = g(\sigma_n) d_n$. The sequence is therefore governed by the orbit of $\sigma_2 = (a, b)$ under R , together with the product of the step multipliers along it. A step with $g > 1$ is an *amplification*. When $g = 1$, R coincides with the cofactor map $T(\alpha, \beta) = (\beta, \alpha + \beta + 1)$ of [3, Definition 3].

F_n and L_n denote the Fibonacci and Lucas numbers ($F_0 = 0, F_1 = 1, L_0 = 2, L_1 = 1$), extended to all integers by $F_{-n} = (-1)^{n+1} F_n$ and $L_{-n} = (-1)^n L_n$, so $F_{-1} = 1$. We use the following standard identities, valid for all integers. Identities (I1) and (I3) are formulas (15a) and (8) of [11], and (I2) is Cassini's identity [11, formula (29)] written with $F_{n-1} = F_{n+1} - F_n$. Identity (I4) follows from the Euclidean algorithm, since $\gcd(F_{n+1}, F_n) = \gcd(F_n, F_{n-1})$ and $\gcd(L_{n+1}, L_n) = \gcd(L_n, L_{n-1})$, down to $\gcd(F_1, F_0) = \gcd(L_1, L_0) = 1$.

- (I1) $F_a L_b = F_{a+b} + (-1)^b F_{a-b}$;
- (I2) $F_{n+1}^2 - F_{n+1} F_n - F_n^2 = (-1)^n$;
- (I3) $F_{j-1} F_n + F_j F_{n+1} = F_{n+j}$;
- (I4) $\gcd(F_n, F_{n+1}) = \gcd(L_n, L_{n+1}) = 1$.

Let $Q = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, and let I denote the 2×2 identity matrix. For $q \geq 0$, $Q^q = \begin{pmatrix} F_{q-1} & F_q \\ F_q & F_{q+1} \end{pmatrix}$, $\text{tr} Q^q = L_q$ and $\det Q^q = (-1)^q$. The Leonardo numbers $Le_n = 2F_{n+1} - 1$ satisfy $Le_n = Le_{n-1} + Le_{n-2} + 1$ [12]. We write Le_n , where [3] wrote $L(n)$, so that L_n always denotes a Lucas number.

Scope.

- S1. We classify the cycles of R with exactly one amplification per period.
- S2. Only the two-term recurrence above is treated.
- S3. Transient lengths, basins of attraction and non-periodic orbits are not characterized.

3. Linear Form of the Cofactor Dynamics

Shifting the cofactors by one makes every non-amplifying step linear.

Lemma 1. *Let (x_0, x_1) be a state and $x_{j+1} = x_j + x_{j-1} + 1$ for $j \geq 1$. If the first k steps of R from (x_0, x_1) are not amplifications, then $R^j(x_0, x_1) = (x_j, x_{j+1})$ for $0 \leq j \leq k$. Moreover, for all $j \geq 0$,*

$$\begin{pmatrix} x_j + 1 \\ x_{j+1} + 1 \end{pmatrix} = Q^j \begin{pmatrix} x_0 + 1 \\ x_1 + 1 \end{pmatrix}.$$

Proof. If $R^j(x_0, x_1) = (x_j, x_{j+1})$ and $g = 1$, then $R(x_j, x_{j+1}) = (x_{j+1}, x_j + x_{j+1} + 1) = (x_{j+1}, x_{j+2})$. With $y_j = x_j + 1$ the recurrence becomes $y_{j+1} = y_j + y_{j-1}$, which is the matrix identity. ■

For $(x_0, x_1) = (1, 1)$ this gives $x_j = Le_j$, which recovers the Leonardo dynamics of [3, Proposition 3].

Lemma 2. *For every state (α, β) , the image $(\alpha', \beta') = R(\alpha, \beta)$ is a state with $\beta' - \alpha' = (\alpha + 1)/g(\alpha, \beta) \geq 1$. In particular, every state on a cycle of R satisfies $\beta > \alpha$.*

Proof. The multiplier $g = \gcd(\beta, \alpha + 1)$ divides β and $\alpha + 1$, hence also $\alpha + \beta + 1$. Therefore $\beta' - \alpha' = (\alpha + 1)/g$ is a positive integer, and $\gcd(\alpha', \beta') = \gcd(\beta, \alpha + 1)/g = 1$. A state on a cycle is the image of its predecessor. ■

Definition 1. A state σ is *periodic* if $R^q(\sigma) = \sigma$ for some $q \geq 1$. The least such q is its *period*, and $\{\sigma, R(\sigma), \dots, R^{q-1}(\sigma)\}$ is a *cycle*. The *cycle multiplier* G is the product of its q step multipliers. The cycle is *single-amplification* if exactly one of these exceeds 1; then $G = g$ is that step multiplier. Its *normalized state*, denoted σ^* , is the state immediately after the amplification. A cycle *attracts* a seed if the state sequence of the seed eventually enters it.

Lemma 3. (a) *Let (α, β) be a state with $\beta > \alpha$, and let $x_0 = \alpha$, $x_1 = \beta$ and $x_{j+1} = x_j + x_{j-1} + 1$. Then (α, β) is the normalized state of a single-amplification cycle of period q and cycle multiplier g if and only if both of the following hold:*

- (i) $\gcd(x_i, x_{i+1}) = 1$ for $1 \leq i \leq q - 1$;
- (ii) $(x_q, x_{q+1}) = g(\alpha, \beta)$ with $g \geq 2$.

Condition (ii) is equivalent to the cycle equation

$$(gI - Q^q)\mathbf{u} = (g - 1)\mathbf{1}, \quad \mathbf{u} = (\alpha + 1, \beta + 1)^\top, \quad \mathbf{1} = (1, 1)^\top.$$

(b) *If the state σ_{n_0} of a sequence lies on a cycle of period q and cycle multiplier G , then $C(n + q) = G C(n)$ for all $n \geq n_0 - 1$.*

Here, as throughout, step j is the step from $R^j(\alpha, \beta)$. As long as the earlier steps are not amplifications, step j has step multiplier $\gcd(x_{j+1}, x_{j+2})$ (see the proof), so condition (i) says that steps $0, \dots, q - 2$ are not amplifications, and condition (ii) says that step $q - 1$ is an amplification that returns to (α, β) .

Proof. (a) Along non-amplifying steps the states are (x_j, x_{j+1}) by Lemma 1. The step from (x_j, x_{j+1}) has multiplier $\gcd(x_{j+1}, x_j + 1) = \gcd(x_{j+1}, x_{j+2})$, which gives (i).

The last step maps (x_{q-1}, x_q) to $(x_q, x_{q+1})/\gcd(x_q, x_{q+1})$. This equals (α, β) exactly when (ii) holds, and then $\gcd(x_q, x_{q+1}) = g\gcd(\alpha, \beta) = g$. Since x_j is strictly increasing, $g \geq 2$, and no earlier state equals (α, β) , so the period is q .

By Lemma 1, $(x_q + 1, x_{q+1} + 1)^\top = Q^q \mathbf{u}$. So (ii) reads $Q^q \mathbf{u} - \mathbf{1} = g(\mathbf{u} - \mathbf{1})$, which is the cycle equation.

(b) The states repeat with period q , and $d_{n+q} = G d_n$ by the relation $d_{n+1} = g(\sigma_n)d_n$ of Section 2. Hence $C(n) = d_n\beta_n$ and $C(n-1) = d_n\alpha_n$ are both multiplied by G . ■

Lemma 3(b) identifies geometric lock-in, that is, $C(n+q) = G C(n)$ for all large n [3, Definition 4], with entering a cycle; conversely, lock-in makes the states periodic, since the state is unchanged when $C(n-1)$ and $C(n)$ are both multiplied by G . Lemma 3(a) reduces single-amplification cycles to a 2×2 linear system with one arithmetic side condition. This is the matrix-product approach of Greene and Niedzielski [4] to periodic integer solutions of recurrences with conditional division. Here the divisor is a gcd rather than a fixed rational number, the dynamics is affine, and all non-amplifying steps use the one matrix Q , so the cycle equation can be solved in closed form (Proposition 4).

4. The Cycle Equation and the Lucas Constraint

Throughout, $q \geq 1$ and $g \geq 2$ are integers, and
 $D = D_q(g) = \det(gI - Q^q) = g^2 - L_q g + (-1)^q$.

Proposition 4. *We have $D \neq 0$, and the cycle equation has the unique solution*

$$\mathbf{u}^* = (u^*, v^*) = \frac{g-1}{D} (g - F_{q-1}, g + F_{q-2}).$$

Proof. The roots of D are φ^q and ψ^q , where $\varphi, \psi = (1 \pm \sqrt{5})/2$. Their discriminant $L_q^2 - 4(-1)^q = 5F_q^2$ is not a perfect square, so D has no integer root. Cramer's rule with $\text{adj}(gI - Q^q) = \begin{pmatrix} g - F_{q+1} & F_q \\ F_q & g - F_{q-1} \end{pmatrix}$ gives $\mathbf{u}^* = \frac{g-1}{D} (g - F_{q+1} + F_q, g + F_q - F_{q-1})$, which is the stated form. ■

Lemma 5. *Let $N(u, v) = v^2 - uv - u^2$. Then $N(u^*, v^*) = -(g-1)^2/D$. Consequently, if u^* and v^* are integers, then $D \mid (g-1)^2$.*

Proof. Put $A = g - F_{q-1}$ and $B = g + F_{q-2}$. Expand, and use $3F_{q-1} + F_{q-2} = L_q$ together with (I2) for $n = q-2$:

$$B^2 - AB - A^2 = -g^2 + L_q g - (F_{q-1}^2 - F_{q-1}F_{q-2} - F_{q-2}^2) = -D.$$

Hence $N(u^*, v^*) = (g-1)^2(B^2 - AB - A^2)/D^2 = -(g-1)^2/D$. ■

The form N is the Fibonacci norm form, which characterizes consecutive Fibonacci pairs [13]. Lemma 5 fixes the norm of every candidate cycle.

Lemma 6. *If $\mathbf{u}^*$ is the shifted normalized state of a single-amplification cycle, then $D > 0$. Equivalently, $g \geq L_q$ when q is even and $g \geq L_q + 1$ when q is odd.*

Proof. By Proposition 4, $v^* - u^* = (g-1)F_q/D$, and $v^* > u^*$ by Lemma 2. Hence $D > 0$.

If q is even, then $L_q \geq 3$ and $D = g(g - L_q) + 1$. For $2 \leq g \leq L_q - 1$ we have $g(L_q - g) \geq L_q - 1$, so $D \leq 2 - L_q < 0$; for $g \geq L_q$ we have $D \geq 1$.

If q is odd, then $D = g(g - L_q) - 1$. This is at most -1 for $g \leq L_q$, and at least L_q for $g \geq L_q + 1$. ■

Proposition 13 in Section 6 extends this bound to cycles with any number of amplifications.

Theorem 7 (Lucas constraint). *Let (α, β) be the normalized state of a single-amplification cycle of period q and cycle multiplier g .*

(a) *If q is even, then $g = L_q$ and $(\alpha + 1, \beta + 1) = (L_q - 1)(F_{q+1}, F_{q+2})$.*

(b) *If q is odd, then $g = L_q + 1$ and $(\alpha, \beta) = (F_{q+1}, F_{q+2})$.*

Proof. Since $\beta > \alpha$ (Lemma 2), Lemma 3(a) and Proposition 4 give $(\alpha + 1, \beta + 1) = \mathbf{u}^*$. This vector is integral, so $D \mid (g - 1)^2$ by Lemma 5, and $D > 0$ by Lemma 6.

Write $\varepsilon = (-1)^q$ and $g = L_q + t$, with $t \geq 0$ if q is even and $t \geq 1$ if q is odd. Then $D = tg + \varepsilon$, so $t(g - 1) \equiv -(t + \varepsilon) \pmod{D}$. Hence $D \mid (g - 1)^2$ implies $D \mid (t + \varepsilon)^2$.

Let q be even. If $t \geq 1$, then $t(L_q + t) + 1 = D \leq (t + 1)^2$, so $L_q \leq 2$, which is impossible. Hence $t = 0$, $g = L_q$ and $D = 1$. Proposition 4 then gives $u^* = (L_q - 1)(L_q - F_{q-1}) = (L_q - 1)F_{q+1}$ and $v^* = (L_q - 1)(L_q + F_{q-2}) = (L_q - 1)F_{q+2}$.

Let q be odd. If $t \geq 2$, then $t(L_q + t) - 1 = D \leq (t - 1)^2$, so $tL_q \leq 2 - 2t < 0$, which is impossible. Hence $t = 1$, $g = L_q + 1$ and $D = L_q$. Proposition 4 then gives $u^* = F_{q+1} + 1$ and $v^* = F_{q+2} + 1$. ■

Theorem 7 uses only positivity and integrality. Coprimality and condition (i) of Lemma 3 remain to be checked. For $q = 1, \dots, 8$ the candidate states are (1,2), (3,5), (3,5), (29,47), (8,13), (220,356), (21,34) and (1563,2529).

5. Classification and Consequences

As in Lemma 3, steps are counted from the normalized state: step j is the step from $R^j(\sigma^*)$.

Lemma 8 (even periods). *For $q = 2m$, the candidate of Theorem 7(a) is*

$(\alpha, \beta) = L_m (F_{3m+1}, F_{3m+2})$ if m is odd, and $(\alpha, \beta) = F_m (L_{3m+1}, L_{3m+2})$ if m is even.

Hence $\gcd(\alpha, \beta)$ equals L_m or F_m , and it equals 1 only for $q \in \{2, 4\}$.

Proof. By (II) with $b = 2m$, we have $L_{2m}F_{2m+1} = F_{4m+1} + 1$ and $L_{2m}F_{2m+2} = F_{4m+2} + 1$.

Hence $\alpha = F_{4m+1} - F_{2m+1}$, $\beta = F_{4m+2} - F_{2m+2}$.

Again by (I1), for $s = 1, 2$,

$$L_m F_{3m+s} = F_{4m+s} + (-1)^m F_{2m+s}, \quad F_m L_{3m+s} = F_{4m+s} - (-1)^m F_{2m+s}.$$

The first expression equals the candidate when m is odd, and the second when m is even. By (I4), the gcd is therefore L_m or F_m . Finally, $L_m = 1$ only for $m = 1$, and $F_m = 1$ only for $m \leq 2$. ■

For example, $q = 6$ gives (220,356) with $\gcd L_3 = 4$, and $q = 8$ gives (1563,2529) with $\gcd F_4 = 3$.

Lemma 9 (odd periods). *For odd $q \geq 3$, step 0 or step 1 from (F_{q+1}, F_{q+2}) is an amplification. Hence no single-amplification cycle has odd period $q \geq 3$.*

Proof. While no amplification occurs, Lemma 1 and (I3) give $x_j = F_{q+j+1} + F_{j+1} - 1$. Step 0 has multiplier $\gcd(F_{q+2}, F_{q+3} + 1)$. If this equals 1, then step 1 has multiplier $\gcd(F_{q+3} + 1, F_{q+4} + 2) = \gcd(F_{q+3} + 1, F_{q+2} + 1)$.

Let $w = q + 2$, which is odd and at least 5. By (I1),

$$\begin{aligned} F_{4k+1} + 1 &= F_{2k+1} L_{2k}, & F_{4k+2} + 1 &= F_{2k+2} L_{2k}, \\ F_{4k+3} + 1 &= F_{2k+1} L_{2k+2}, & F_{4k+4} + 1 &= F_{2k+1} L_{2k+3}. \end{aligned}$$

So $L_{2k} \geq 3$ divides $\gcd(F_w + 1, F_{w+1} + 1)$ if $w = 4k + 1$, and $F_{2k+1} \geq 2$ divides it if $w = 4k + 3$; in both cases $k \geq 1$. Thus step 0 or step 1 is an amplification. Since $q \geq 3$, this contradicts Lemma 3(a)(i). ■

The quantity $\gcd(F_n + 1, F_{n+1} + 1)$ is the classical unbounded case among gcds of shifted Fibonacci numbers [14–16]. Lemma 9 shows that it is exactly what rules out the odd candidates.

Theorem 10 (Main Theorem). *The cycles of R with exactly one amplification per period are precisely the following:*

- (i) the fixed point (1,2), with cycle multiplier 2;
- (ii) the 2-cycle (3,5) $\rightarrow$ (5,9), with cycle multiplier 3;
- (iii) the 4-cycle (29,47) $\rightarrow$ (47,77) $\rightarrow$ (77,125) $\rightarrow$ (125,203), with cycle multiplier 7.

Accordingly, a sequence whose states reach one of these cycles eventually satisfies $C(n+1) = 2C(n)$, $C(n+2) = 3C(n)$ or $C(n+4) = 7C(n)$, respectively.

Proof. By Theorem 7 and Lemmas 8 and 9, only $q \in \{1, 2, 4\}$ remain, with the candidates (1,2), (3,5) and (29,47). Each is a single-amplification cycle:

- $R(1,2) = (1,2)$, with step multiplier $\gcd(2,2) = 2$.
- From (3,5) the step multipliers are $\gcd(5,4) = 1$ and $\gcd(9,6) = 3$.

- From (29,47) they are $\gcd(47,30) = \gcd(77,48) = \gcd(125,78) = 1$ and $\gcd(203,126) = 7$, and $(203,329)/7 = (29,47)$.

The final statement is Lemma 3(b). ■

Table 1 records the invariants of the three cycles. All have $D = 1$.

Table 1. The three single-amplification cycles of Theorem 10, with their normalized states (α, β) and shifted states $(u, v) = (\alpha + 1, \beta + 1)$. In each case $N(u, v) = -(g - 1)^2$.

q	g	(α, β)	(u, v)	D	$N(u, v)$
1	$2 = L_1 + 1$	(1,2)	$(2,3) = (F_3, F_4)$	1	-1
2	$3 = L_2$	(3,5)	$(4,6) = 2(F_3, F_4)$	1	-4
4	$7 = L_4$	(29,47)	$(30,48) = 6(F_5, F_6)$	1	-36

Corollary 11. (a) The fixed point (1,2) is the only cycle of period 1, and $(3,5) \rightarrow (5,9)$ is the only single-amplification cycle of period 2.

(b) Every cycle of period at most 2, with any number of amplifications, is one of these two.

(c) [3, Corollary 5] states that $(1,2)$ and $(3,5) \leftrightarrow (5,9)$ are the only coprime periodic orbits. By (b) this holds for periods $q \leq 2$; by Theorem 10(iii) it fails in general.

Proof. (a) A non-amplifying step never fixes a state, so every cycle of period 1 is single-amplification; now apply Theorem 10.

(b) A cycle of period 2 needs at least one amplification, since otherwise the second coordinate increases. Suppose both steps amplify, with multipliers $g_1, g_2 \geq 2$, and put $P = g_1 g_2$. Returning to (α, β) requires $\alpha + \beta + 1 = P\alpha$ and $\beta/g_1 + g_2\alpha + 1 = g_2\beta$. The first relation gives $\beta = (P - 1)\alpha - 1$. Substituting this into the second relation, multiplied by g_1 , gives $\alpha(P^2 - 3P + 1) = P - 1 + g_1 \leq 3/2 P - 1$. Hence $(2P - 1)(P - 4) \leq 0$, so $P = 4$, $\alpha = 1$ and $\beta = 2$. This is the fixed point traversed twice.

(c) is immediate. ■

Parts (a) and (b) give a purely algebraic proof of [3, Theorems 6 and 8]; the original argument for Theorem 8 relies in part on computation [17]. The correction notice to [3] now restricts [3, Corollary 5] to periods 1 and 2, in agreement with (c).

Boundary cases. Theorem 10 and Lemma 3(b) recover the known regimes.

- Seeds (1,1): the state at index 4 is (3,5), so $C(n + 2) = 3 C(n)$ for $n \geq 3$. This is the closed form of A083658 [1].
- Seeds (1, p) with p even: the state at index 4 is (1,2), so $C(n + 1) = 2 C(n)$ for $n \geq 3$ [2, Theorem 1].

- Seeds $(1, p)$ with p odd and prime to 3: the state at index 6 is $(3, 5)$, so $C(n + 2) = 3 C(n)$ for $n \geq 5$ [2, Theorems 2 and 3].

These statements agree with the values tabulated in [2, Table 3] for $p \in \{1, 2, 5, 7, 10\}$ and in [3, Table 2] for odd $p \leq 49$ prime to 3 (with the column labels corrected below), and we checked them directly for all $p \leq 49$.

Proposition 12. *For $p \in \{9, 15, 27\}$, the sequence C_p with seeds $(1, p)$ satisfies $C_p(n_p + 4k + r) = 7^k d_p c_r$ ($k \geq 0, r = 0, 1, 2, 3$), where $(c_0, c_1, c_2, c_3) = (29, 47, 77, 125)$ and $(n_p, d_p) = (25, 3^9), (6, 3)$ and $(14, 3^5)$, respectively. In particular, $C_p(n + 4) = 7 C_p(n)$ for all $n \geq n_p$, and n_p is the least index with this property.*

Proof. Computing $C_p(1), \dots, C_p(n_p + 1)$ from the recurrence gives $d_{n_p+1} = d_p$ and $\sigma_{n_p+1} = (29, 47)$. The next states follow the 4-cycle with constant scale until its amplifying step. Theorem 10(iii) and Lemma 3(b) give the rest. Minimality is checked directly at $n_p - 1$. ■

For example, $C_{15}(1), \dots, C_{15}(7)$ are 1, 15, 17, 33, 51, 87 and 141, and $\gcd(87, 141) = 3$ gives $\sigma_7 = (29, 47)$ with $d_7 = 3$.

In [2, Section 4.3] these seeds were described as showing non-geometric transients through $n = 30$; the correction notice to [2] now records the lock-in of Proposition 12. The earlier description has a simple cause: the growth is exactly geometric with period four, so the one-step ratios $C_p(n + 1)/C_p(n)$ cycle through four values instead of settling, while the four-step ratio is exactly 7 (Figure 1).

The other odd multiples of 3 below 50 behave as follows: $p = 3$ and $p = 21$ reach $(3, 5)$; $p = 33$ and $p = 39$ reach $(29, 47)$; and $p = 45$ shows no repeated state within 1,200 steps. The seed $(1, 87)$ reaches the period-22 cycle described below.

For the record, the recurrence gives $C_{21}(9) = 567$ and $C_{21}(10) = 945$, and $C_{27}(7), \dots, C_{27}(10)$ equal 237, 387, 627 and 1017. These values correct [2, Table 4], and the columns of [3, Table 2] labelled $C(5)$ and $C(6)$ list $C(6)$ and $C(7)$. Both corrections are recorded in the correction notices to [2] and [3], and no theorem of either paper is affected.

How common is the new cycle? We took all 8,771 coprime seeds (a, b) with $1 \leq a, b \leq 120$ (box 120), iterated R for at most 1,200 steps, and recorded the cycle reached, detected as a repeated state. We repeated the census for the 24,463 coprime seeds with $1 \leq a, b \leq 200$ (box 200). Table 2 gives both distributions in full; the archived code [18] reproduces them. The 4-cycle attracts about as many seeds as the 2-cycle in box 120 and the most seeds in box 200. Together, the three single-amplification cycles attract 7,291 seeds (83.1%) in box 120 and 19,114 seeds (78.1%) in box 200. Besides the period-22 cycle described below, one further cycle appears, of period 65, reached by two seeds of box 200. The counts are unchanged in both boxes when the step limit is doubled to 2,400. The shares do depend on the box: in the larger box the

fixed point and the 2-cycle lose ground, and the share of seeds with no detected cycle rises from 10.0% to 13.4%.

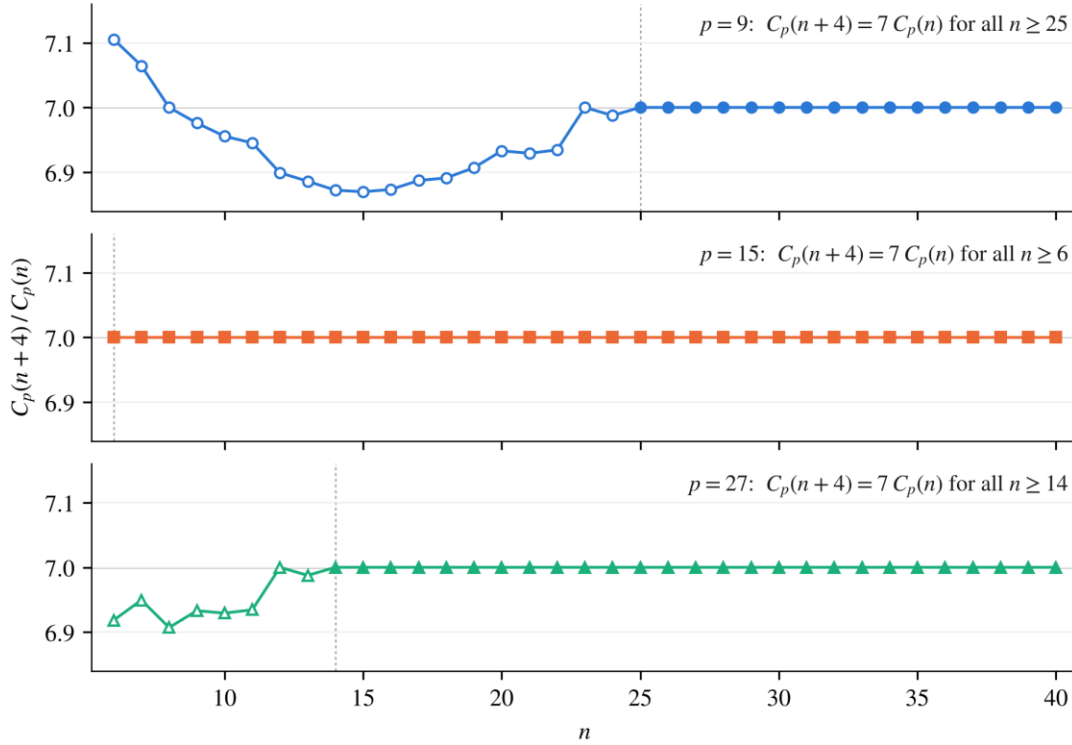


Figure 1. Four-step ratio $C_p(n+4)/C_p(n)$ for the seeds $(1, p)$ with $p = 9, 15, 27$. Hollow markers precede the lock-in index n_p (dotted line); from n_p on the ratio is exactly 7 (Proposition 12). For $6 \leq n < n_p$ the ratio equals 7 only at isolated indices ($p = 9$: $n = 8$ and $n = 23$; $p = 27$: $n = 12$), which are the hollow markers at 7.0.

Table 2. Limit behaviour of the coprime seeds (a, b) in box 120 ($1 \leq a, b \leq 120$) and box 200 ($1 \leq a, b \leq 200$), after at most 1,200 steps of R .

Limit behaviour	Box 120: seeds	Share	Box 200: seeds	Share
Fixed point (1,2): $\times 2$ per step	912	10.4%	1,810	7.4%
2-cycle (3,5): $\times 3$ per two steps	3,226	36.8%	7,918	32.4%
4-cycle (29,47): $\times 7$ per four steps	3,153	35.9%	9,386	38.4%
Period-22 cycle through (97,157)	607	6.9%	2,071	8.5%
Period-65 cycle through (4003,6477)	0	0.0%	2	<0.1%
No repeated state within 1,200 steps	873	10.0%	3,276	13.4%
Total	8,771	100%	24,463	100%

The seeds with no detected cycle do not stay small. For the 873 such seeds of box 120, the cofactor β of the state has between 27 and 253 digits (median 102) after 1,200 steps, and between 39 and 504 digits (median 353) after 2,400 steps; for the seed (1,45) it grows from 40 to 211 digits. These orbits keep growing over the range computed, although no seed is proved to diverge.

Hypothesis S1 cannot be dropped. The seed (1,87) reaches a cycle of period 22 through the state (97,157). This cycle has six amplifications, with step multipliers 3,3,3,3,33 and 15, so its cycle multiplier is $40,095 = 3^6 \cdot 5 \cdot 11$. The seeds (3,187) and (187,191) reach a second such cycle, of period 65 through the state (4003,6477), with 19 amplifications and cycle multiplier $3^{15} \cdot 5^3 \cdot 7^2 \cdot 19 \cdot 23 = 38,406,643,198,875$. These are further coprime periodic orbits, so Theorem 10 does not classify all cycles.

6. Discussion

The three cycle multipliers $2 = L_1 + 1$, $3 = L_2$ and $7 = L_4$ come from a single mechanism. Between amplifications the shifted cofactors evolve by the Fibonacci matrix. A single-amplification cycle closes only if the norm of its shifted state equals $-(g-1)^2/D$ with $D > 0$ dividing $(g-1)^2$. For even periods this forces $D = 1$ and leaves only the Lucas multipliers; gcd identities for Fibonacci and Lucas numbers then exclude all periods except 1, 2 and 4. The arithmetic of the golden field, which [3] detected through the Pell equation $x^2 - 5y^2 = 4$, reappears here as the Fibonacci norm form.

Several amplifications. The linearization of Section 3 does not depend on the number of amplifications. Every cycle contains an amplification, because β increases along non-amplifying steps. Composing the affine steps around a cycle extends Lemma 3(a) and Lemma 6 as follows.

Proposition 13. *Let (α, β) be a state of a cycle of period q that immediately follows an amplification, and let the cycle have $k \geq 1$ amplifications. Counted from (α, β) , let the i -th amplification be the last step of a segment of $q_i \geq 1$ steps and have step multiplier $g_i \geq 2$, so that $q_1 + \dots + q_k = q$. Put $G = g_1 \dots g_k$, $s_i = q_{i+1} + \dots + q_k$ and $\mathbf{u} = (\alpha + 1, \beta + 1)^\top$. Then*

$$(GI - Q^q) \mathbf{u} = \sum_{i=1}^k g_1 \dots g_{i-1} (g_i - 1) (F_{s_i+1}, F_{s_i+2})^\top.$$

Consequently, each pattern $(q_1, \dots, q_k; g_1, \dots, g_k)$ belongs to at most one cycle, and $G > \varphi^q$; equivalently, $G \geq L_q$ if q is even and $G \geq L_q + 1$ if q is odd.

Proof. A step from (α, β) with step multiplier g gives $(\beta/g, (\alpha + \beta + 1)/g)$, whose shifted form is $((\beta + 1) + g - 1, (\alpha + 1) + (\beta + 1) + g - 1)/g = (Q\mathbf{u} + (g-1)\mathbf{1})/g$. A segment of q_i steps whose last step alone amplifies therefore maps $\mathbf{u}$ to $(Q^{q_i}\mathbf{u} + (g_i - 1)\mathbf{1})/g_i$. Composing the k segments and multiplying by G gives $G\mathbf{u} = Q^q\mathbf{u} + \sum_i g_1 \dots g_{i-1} (g_i - 1) Q^{s_i}\mathbf{1}$, and $Q^s\mathbf{1} = (F_{s+1}, F_{s+2})^\top$. Since $\det(GI - Q^q) = D_q(G) \neq 0$ (Proposition 4), $\mathbf{u}$ is determined by the pattern. The row vector $\mathbf{w} = (1, \varphi)$ satisfies $\mathbf{w}Q = \varphi \mathbf{w}$, so multiplying the cycle equation by $\mathbf{w}$ gives $(G - \varphi^q) \mathbf{w}\mathbf{u} = \mathbf{w}\mathbf{c}$, where $\mathbf{c}$ is the right-hand side. Both $\mathbf{w}\mathbf{u}$ and $\mathbf{w}\mathbf{c}$ are positive, so $G > \varphi^q$. Finally, $\varphi^q = L_q - \psi^q$, where $0 < \psi^q < 1$ for even q and $-1 < \psi^q < 0$ for odd q . ■

For $k = 1$ the right-hand side is $(g-1)\mathbf{1}$, which gives Lemma 3(a), and the bound is Lemma 6. For the cycles of periods 22 and 65 in Section 5, $G/\varphi^q \approx 1.012$ and 1.0005 : the cycle multiplier must exceed, and here barely exceeds, the growth factor φ^q of the Fibonacci steps. This mirrors

the cycle equations of the $3x + 1$ map [8, 10], where the power of 2 removed along a positive cycle must exceed the power of 3 introduced.

What Proposition 13 does not provide is the divisibility behind Theorem 7. For $k = 1$ the right-hand side is a multiple of $\mathbf{1}$, which fixes the norm of $\mathbf{u}$ (Lemma 5) and forces $D = 1$ or $D = L_q$. For $k \geq 2$ the norm is not fixed, and D need not be small: $D = 19,726,741$ for the period-22 cycle. Each pattern carries at most one cycle, but the patterns are not bounded in advance, so classifying even the cycles with two amplifications needs a new bound; this is the analogue of the m -cycle problem [10].

The limitations follow the scope labels.

- (S1) Cycles with several amplifications exist, as the cycles of periods 22 and 65 in Section 5 show. Proposition 13 reduces each amplification pattern to one linear system, but the classification of these cycles is open.
- (S2) GCD-augmented recurrences of higher order are not treated. The reduction itself carries over. For the Tribonacci analogue $C(n) = C(n-1) + C(n-2) + C(n-3) + \gcd(C(n-1), C(n-2), C(n-3))$, dividing consecutive triples by their gcd gives a map on coprime triples (α, β, γ) with step multiplier $\gcd(\beta, \gamma, \alpha + 1)$, and the shift $x \mapsto 2x + 1$ makes every step affine in the Tribonacci matrix. Proposition 13 and its proof then carry over, with the Tribonacci matrix Q_3 and the Tribonacci constant in place of Q and φ , and with $Q_3^{s_i} \mathbf{1}$ in place of $(F_{s_i+1}, F_{s_i+2})^\top$. Proposition 4 is not needed there: G is not an eigenvalue of Q_3^q , since the eigenvalues are τ^q , which is irrational for the Tribonacci constant τ , and two numbers of modulus less than 1; and Q_3 has a positive left eigenvector by the Perron–Frobenius theorem. For example, $(1, 2, 4)$ is a fixed point with step multiplier 2. What does not carry over is the quadratic norm form of Lemma 5, on which Theorem 7 rests.
- (S3) We characterize periodic behaviour only. In Table 2, 10.0% of the seeds of box 120 and 13.4% of those of box 200 show no repeated state within 1,200 steps, so the share grows with the box; these orbits keep growing over the steps computed (Section 5).

Whether every seed eventually reaches a cycle is open. The question resembles convergence problems for Collatz-type maps [7] and for other nonlinear integer recurrences [4–6].

For the anomalous class of [2] (odd p divisible by 3), Proposition 12 and the examples after it show that seeds split among the 2-cycle, the 4-cycle, the period-22 cycle and seeds with no detected cycle. This sharpens Open Problem 1 of [2].

7. Conclusions

Exactly three cycles of the gcd-augmented Fibonacci cofactor map have a single amplification per period: $(1, 2), (3, 5) \rightarrow (5, 9)$, and the new 4-cycle $(29, 47) \rightarrow (47, 77) \rightarrow (77, 125) \rightarrow (125, 203)$. Their cycle multipliers are 2, 3 and 7. The 4-cycle corrects [3, Corollary 5], explains the anomalous seeds 9, 15 and 27 of [2], and attracts about as many seeds as the classical 2-cycle in the smaller census and more in the larger one. For any number of amplifications, the same

linearization gives a cycle equation and the bound $G > \varphi^q$ (Proposition 13). Two questions remain open: the classification of multi-amplification cycles, which is the analogue of the m -cycle problem for the $3x + 1$ map, and whether every seed reaches a cycle.

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Every numerical result, both tables and Figure 1 can be reproduced with the Python script `reproduce.py`, version 2.0.0, archived on Zenodo with a persistent identifier [18]; the source is maintained at <https://github.com/mhawarey/single-amplification-cycles-gcd-fibonacci>.

References

- [1] Hanna, P. D. (2003). Sequence A083658: $a(n) = a(n-1) + a(n-2) + \gcd(a(n-1), a(n-2))$. *The On-Line Encyclopedia of Integer Sequences*. Retrieved 9 October 2026, from <https://oeis.org/A083658>
- [2] Hawarey, M. (2026). GCD-augmented Fibonacci sequences: Closed forms for generalized seeds. *AIR Journal of Mathematics and Computational Sciences*, 2026, Article AIRMCS2026211 (correction notice issued 9 October 2026). <https://doi.org/10.65737/AIRMCS2026211>
- [3] Hawarey, M. (2026). Cofactor dynamics and Pell equations in GCD-augmented Fibonacci recurrences: The algebraic structure of geometric stabilization. *AIR Journal of Mathematics and Computational Sciences*, 2026, Article AIRMCS2026571 (correction notice issued 9 October 2026). <https://doi.org/10.65737/AIRMCS2026571>
- [4] Greene, J., & Nieldzielski, K. (2010). Periodic solutions to some difference equations over the integers. *Journal of Difference Equations and Applications*, 16(4), 321–346. <https://doi.org/10.1080/10236190802395172>
- [5] Greene, J. (2008). The unboundedness of a family of difference equations over the integers. *The Fibonacci Quarterly*, 46/47(2), 146–152. <https://doi.org/10.1080/00150517.2008.12428173>
- [6] Kirschenhofer, P., Pethő, A., & Thuswaldner, J. M. (2008). On a family of three term nonlinear integer recurrences. *International Journal of Number Theory*, 4(1), 135–146. <https://doi.org/10.1142/S1793042108001250>
- [7] Lagarias, J. C. (1985). The $3x + 1$ problem and its generalizations. *The American Mathematical Monthly*, 92(1), 3–23. <https://doi.org/10.2307/2322189>

- [8] Böhm, C., & Sontacchi, G. (1978). On the existence of cycles of given length in integer sequences like $x_{n+1} = x_n/2$ if x_n even, and $x_{n+1} = 3x_n + 1$ otherwise. *Atti della Accademia Nazionale dei Lincei, Rendiconti, Classe di Scienze Fisiche, Matematiche e Naturali*, 64, 260–264.
- [9] Steiner, R. P. (1978). A theorem on the Syracuse problem. In *Proceedings of the Seventh Manitoba Conference on Numerical Mathematics and Computing* (Winnipeg, 1977), *Congressus Numerantium* 20 (pp. 553–559). Utilitas Mathematica.
- [10] Simons, J. L., & de Weger, B. M. M. (2005). Theoretical and computational bounds for m -cycles of the $3n+1$ -problem. *Acta Arithmetica*, 117(1), 51–70.
<https://doi.org/10.4064/aa117-1-3>
- [11] Vajda, S. (1989). *Fibonacci & Lucas numbers, and the golden section: Theory and applications*. Ellis Horwood.
- [12] Catarino, P. M. M. C., & Borges, A. (2020). On Leonardo numbers. *Acta Mathematica Universitatis Comenianae*, 89(1), 75–86.
<http://www.iam.fmph.uniba.sk/amuc/ojs/index.php/amuc/article/view/1005>
- [13] Jones, J. P. (1975). Diophantine representation of the Fibonacci numbers. *The Fibonacci Quarterly*, 13(1), 84–88.
<https://doi.org/10.1080/00150517.1975.12430693>
- [14] Chen, K.-W. (2011). Greatest common divisors in shifted Fibonacci sequences. *Journal of Integer Sequences*, 14, Article 11.4.7.
<https://cs.uwaterloo.ca/journals/JIS/VOL14/Chen/chen70.html>
- [15] Rahn, A., & Kreh, M. (2018). Greatest common divisors of shifted Fibonacci sequences revisited. *Journal of Integer Sequences*, 21, Article 18.6.7.
<https://cs.uwaterloo.ca/journals/JIS/VOL21/Kreh/kreh6.html>
- [16] Hieu, L. D., Spilker, J., & Thang, L. B. (2022). The greatest common divisor of shifted Fibonacci numbers. *Journal of Integer Sequences*, 25, Article 22.1.7.
<https://cs.uwaterloo.ca/journals/JIS/VOL25/Thang/thang42.html>
- [17] Hawarey, M. (2026). Verification code for "Cofactor Dynamics and Pell Equations in GCD-Augmented Fibonacci Recurrences: The Algebraic Structure of Geometric Stabilization" (Version v2.0.0) [Computer software]. Zenodo.
<https://doi.org/10.5281/zenodo.23282465>
- [18] Hawarey, M. (2026). Reproduction code for "Exactly three single-amplification cycles in GCD-augmented Fibonacci recurrences" (Version v2.0.0) [Computer software]. Zenodo.
<https://doi.org/10.5281/zenodo.23266649>