PAPER EVALUATION REPORT

AIR Journals — AI-Powered Academic Review

(Under Strict Scholarly Supervision)

Submission ID	***AIR-2026-000983***
Paper Title	Escape from Geometric Lock-in in GCD-Augmented Fibonacci Recurrences
Author	Mosab Hawarey
Submission Date	23 October 2026
Journal	AIR Journal of Mathematics and Computational Sciences (DOI: 10.65737/AIRMCS) (ISSN: 3142-7197)
Review Completed	26 October 2026
Reviewer	AIR Journals AI Review System
Report Version	Final Evaluation Report

References are to sections, numbered results, tables and figures, which do not depend on pagination.

EXECUTIVE SUMMARY

Overall Score: 80/100

Editorial Decision: Minor Revisions

Decision Rationale: This is a correct and independently reproducible paper. It gives the first proofs that seeds of this recurrence can escape geometric lock-in, together with a finite certificate that decides whether a stretch without amplifications is permanent. We verified every proof and reproduced the theorems and both census tables independently. Minor revisions are needed:

- correct one statement that conflicts with the paper's own three-fate framework (Section 2);
- label every census figure with the protocol that produced it, and correct the caption of Table 1;
- document the primality standard and software behind the computational certificates;
- broaden the literature context and make two small citation corrections.

Primary Strengths:

- **Correct results, independently reproduced.** Theorems 6 and 7, all seven rows of Table 1 and both boxes of Table 2 were reproduced exactly by an implementation written from the manuscript text alone.
- **A finite certificate for an infinite-time question.** Lemmas 1–2 and Theorem 3 reduce "will this seed ever be amplified again?" to orbit tests modulo the finitely many primes dividing $N_0^2 - 1$. Section 3 shows why no step limit can replace it: one state first amplifies at step 47,359.
- **Honest scope and limitations.** Scope labels S1–S3 (Section 2) are revisited one by one in Section 6, fate (c) is kept explicitly open, and the census is presented as a lower bound.

Primary Concerns:

- **One incorrect statement.** "Escape is therefore exactly the failure of geometric lock-in" (Section 2) contradicts Definition 1, because seeds of fate (c) also fail to lock in.
- **Census reporting.** The Abstract and Conclusions report 956 certified escapes and 2,320 undecided seeds, while Table 2 reports 907 and 2,369, without saying that

these come from different protocols. The caption of Table 1 ("that escape") implies a complete list, although several seeds in its range remain undecided.
- **Computational trust and context.** The primality standard and software versions behind the certificates are not stated, and the literature on orders of φ modulo primes (Artin-type problems), which bears directly on Proposition 4 and Conjecture 9, is not cited.

Timeline for Resubmission: Less than 1 week (if AI-assisted); 2–4 weeks otherwise

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COMPREHENSIVE EVALUATION RESULTS

Citation Integrity Verification (Step 1)

Status: **PASS**

Verification Level: 100% — Every citation checked

Citation Statistics:
- Total bibliography entries: 17
- Bibliography entries cited in text: 17 (100%)
- Reference numbers cited in text that have bibliography entries: 17 of 17 (100%)
- References confirmed as genuine publications against publisher, journal, archive and database records: 17 of 17, with two small gaps in bibliographic details (listed below)

Issues Identified:

Uncited Bibliography Entries (Padding): Count: 0 (0%) — Severity: None

Missing Bibliography Entries: Count: 0 — Severity: None

Additional Citation Issues:
- **Formatting consistency:** Good. Numbered citations, including ranges and precise pinpoints such as [4, Lemma 3(b)], with APA-style entries used consistently. Minor completeness items are listed below.
- **Self-citation rate:** 23.5% (4 of 17: [2], [3], [4] and the code archive [17]). Justified, because the paper answers an open question from the author's own series, and clearly disclosed ("The author's earlier work [2–4]", Section 1).
- **Figure/table attribution:** Complete. Figure 1 and Tables 1–2 are original; the cycle rows of Table 2 are explicitly identified as reproducing [4, Table 2].
- **Data source citations:** One gap. The software PARI, used for factoring in the robustness check (Section 5), is not cited and its version is not given.

Required Citation Corrections:

Critical (Must Fix): None

Important (Should Fix):
1. Section 5: cite PARI/GP with the version used.
2. References [2] and [3] carry correction notices (issued 9 October, updated 21 October 2026). State which version is cited, since several pinpoints depend on it: [2, Theorem 4], [2, Section 4.3], [3, Theorem 5 and Lemma 4], [3, Proposition 3] and [3, Section 6.5].

Minor (Recommended):
1. [12]: add the volume editors (J. Sander, J. Steuding and R. Steuding) and the full book title, as is already done for the editor of [6].
2. [7]: give the year as 2008/2009, matching the combined volume 46/47.

3. [6]: add the DOI or ISBN of the published AMS volume alongside the arXiv link; [16]: use an https link.
4. [2]–[4]: avoid repeating the year as the volume ("2026, Article …").
5. [17]: add the GitHub URL to the entry, or add a short "Code availability" statement.

Citation Integrity Conclusion: Citation integrity is excellent. Every bibliography entry is cited, every citation has an entry, and every reference is genuine. The two important corrections are small improvements in attribution and version control.

Plagiarism Content Assessment (Step 2)

Status: **PASS**

Tool Used: Premium Professional Service

Similarity Scores:

- Raw similarity score: 6%
- Adjusted similarity score (after removing legitimate matches): <1%
- Assessment: Green (0–14%)

Analysis: All 20 matches were classified as benign:

- 16 matches: bibliographic reference entries, i.e. titles and journal names of properly cited works
- 2 matches: boilerplate declarations (the AI-assistance disclosure wording, and the conflict-of-interest and funding statement)
- 1 match: a common phrase ("in the census of")
- 1 match: the opening sentence of the Abstract, which closely follows the abstract of the author's own paper [4]

Apart from these, there are no matches anywhere in the main text, including the results sections (Sections 4–5).

Self-Plagiarism Check:

- Author's prior work matched: Yes, one definitional sentence of the Abstract (about 28 words) and the bibliographic metadata of [2]–[4]
- Overlap: less than 1% of the manuscript
- Properly disclosed: Yes. The earlier papers are identified as the author's own (Section 1), and the notation of [4] is adopted explicitly (Section 2).
- Properly self-cited: Yes, precisely and throughout
- Assessment: Acceptable

Required Corrections: None. Rewording the first sentence of the Abstract is suggested under Minor Issues.

Plagiarism Conclusion: No plagiarism detected. The text is original, including where the paper builds directly on the author's earlier papers.

Quality Evaluation Results (Step 3)

Overall Quality Score: **80/100**

Score Breakdown by Category:

CATEGORY A: CONCEPTUAL FOUNDATION

Score: 22/30

Evaluation:

The paper answers a precisely posed question. Whether every seed reaches a cycle was left open in [4, Section 6], where 10.0% of the coprime seeds up to 120 and 13.4% of those up

to 200 showed no repeated state, with no proof either way. The paper answers it decisively and in the negative, and Section 3 explains convincingly why the question needs a proof rather than more computation. The wider significance is less developed. The gap is documented only within the author's own series, and the analogy with Collatz-type divergence is suggestive rather than demonstrated: as Section 6 itself notes, this map is linear between amplifications, which is exactly what makes a certificate possible here and not for the 5x + 1 map.

Literature use is accurate and appropriately modest. The relation to known results is stated precisely: the divisibility in Lemma 2 is "a special case of the bounds of Chen and Pan [9, Example 6]", and part (c) is "the analogue of the classical unbounded case" of [11–14]. Two relevant bodies of work are missing. The "full primes" of Proposition 4, where φ has the maximal order 2(r + 1) modulo r, belong to the literature on Fibonacci primitive roots and Artin's primitive root conjecture. That literature is also the natural starting point for the "density heuristic for these orders" that Section 6 calls for. Other gcd-driven integer sequences would provide useful comparison points in Section 1. Finally, several basic facts are imported from [2]–[4] without justification, and two of those papers carry correction notices.

The exposition is precise. Every new notion is defined before use: state, step multiplier, amplification, cycle multiplier, shifted state, norm, golden integer, escape, the three fates and full primes. The scope labels S1–S3 delimit every claim. Two points need attention:

- Step and index conventions are mixed. Steps are counted from 0 ("step j is the step from $R^j(\sigma)$"), states are indexed σ_n from n = 2, and Figure 1 says that (1, 135) "escapes after step 10", while Theorem 6(b) places the last amplification at step 9.
- Census figures from two protocols appear without labels: 907 certified escapes and 2,369 undecided seeds in Table 2 (1,200 steps, 30-digit cap), against 956 and 2,320 in the Abstract and Conclusions (the extended check in Section 5).

Key Strengths:

- A precisely posed open question, answered decisively, with a clear argument for why proof is needed
- Precise definitions and scope labels that make every claim easy to delimit

Areas for Improvement:

- Cite the literature on Fibonacci primitive roots and Artin-type densities at Proposition 4 and Conjecture 9
- Mention one or two other gcd-driven recurrences in Section 1, and state which kinds of recurrence the certificate idea could transfer to (Section 6)
- Give one-line justifications for the facts imported from [2]–[4] (for example $C_{ka,kb} = kC_{a,b}$ and the cofactor factorization)
- Fix one step and index convention, state it once in Section 2, and align Figure 1 and Theorem 6(b) with it
- Label every census figure with its protocol

CATEGORY B: METHODOLOGICAL RIGOR

Score: 26/30

Evaluation:

The method fits the question exactly. The reduction runs as follows:

- Between amplifications the shifted state evolves by the Fibonacci matrix and its golden integer is multiplied by φ (Lemma 1).
- A prime can trigger an amplification only if the shifted state is (0, 1) modulo that prime, which forces it to divide $N_0^2 - 1$ (Lemma 2).
- Periodicity modulo each such prime then makes the question finite (Theorem 3), and Theorem 7 obtains infinitely many escaping seeds by an explicit construction rather than by computation.

Execution is very strong. We checked every proof and re-implemented the method from the manuscript text alone, without the author's code. The independent computation reproduced:

- Theorem 6: the closed forms for the seeds (11, 5) and (1, 135), the roots ρ = 105 and ρ = 14, the primitive root modulo 179 and the order 45 modulo 181
- Theorem 7 for t = 1, …, 2,000
- all seven rows of Table 1, and every count in Table 2 for both boxes, unchanged at 2,400 steps
- the 54,795 states of Section 3, including the ten late amplifications and the latest one at step 47,359
- the full primes below 100 and the 489 below 10^4; the shares of certified escapes for boxes 50–200; the m-family counts; and the 43 escaping seeds (1, p) with p ≤ 2,000 under the stated computational caps

Three technical points need correction. The proof of Proposition 4(a) cites Lemma 1 for the norm of the residues (F_j, F_{j+1}), but Lemma 1 concerns the shifted states of actual states and (0, 1) is not one; Cassini's identity gives the fact directly. The remark after Theorem 3 applies the theorem to the fixed point (1, 2), whose norm is −1, outside its hypothesis $N_0^2 \neq 1$. And the primality standard for the factors used in the census certificates is not stated, although a certificate is valid only if every factor is genuinely prime. The explicit theorems are unaffected, since they use small primes that are easily proved prime.

Reproducibility is excellent. All computational parameters are stated: box sizes, step limits, the 30-digit factoring cap, iteration for $r \leq 3 \cdot 10^6$ and discrete logarithms for split primes up to 10^{12}. We confirmed that the code is archived publicly on Zenodo (version 1.0.0, MIT licence) and on GitHub, with reproduce.py. The remaining gaps are minor:

- software versions (Python, PARI/GP) and the primality setting are not given;
- the extended check is described less precisely than the main census, for example how the 612 and 2,222 seeds are proved to amplify again after step 2,400.

Key Strengths:

- An elegant reduction (linearization, norm form, periodicity modulo r) matched exactly to the question
- Correct, explicit proofs and fully reproducible computations

Areas for Improvement:

- Proof of Proposition 4(a): cite Cassini's identity instead of Lemma 1
- Section 3: restrict the remark on the three cycles to (3, 5) and (29, 47); the fixed point (1, 2) is covered by Lemma 2(c)
- Section 5: add a short paragraph on computational details (software and versions, primality standard, extended-check procedure)
- Section 6: add a sentence on what a certificate for fate (c) would have to control

CATEGORY C: EVIDENCE & REASONING

Score: 17/20

Evaluation:

Every main claim is proved: the existence of escaping seeds (Theorem 6), infinitely many of them (Theorem 7), the condition 30 | N for escaping orbits (Proposition 4) and the return of the golden ratio as growth rate (Corollary 5). The argument runs as a clean deductive chain from Lemma 1 to Theorem 7, and the proof of Theorem 6 is a model of short, hand-checkable reasoning. Conjecture 9 is clearly separated from the theorems, and its evidence is stated.

A few statements go further than the paper's own framework or evidence:

- Section 2: "Escape is therefore exactly the failure of geometric lock-in." By Definition 1, both fate (b) and fate (c) fail to lock in, and the paper itself keeps fate (c) open (S2). Escape is one of the two ways lock-in can fail.
- Table 1: the caption ("the seeds (1, p) with p < 400, p odd and divisible by 3, that escape") implies a complete list. Several seeds in that range remain undecided, for example p = 45, which Figure 1 itself describes as undecided (16 of the 67 such seeds in our recomputation at 2,400 steps).
- Sections 1 and 3: "it also explains why lock-in is common". Proposition 4 shows that amplifications are frequent when $30 \nmid N_0$; that this leads to cycles is a heuristic step.
- Counts obtained under computational caps are lower bounds. For example, the (1, p) count of 43 (Section 5) is exact under the stated caps, but a Pohlig–Hellman discrete logarithm beyond 10^{12} also certifies (1, 1419) and (1, 1635) as escaping, which we confirmed independently.

Key Strengths:

- Proof-level evidence for every main claim, with explicit, verifiable witnesses
- A tight deductive structure, with the conjecture clearly separated from the theorems

Areas for Improvement:

- Section 2: replace the sentence on escape, for example by "Escape is one of the two ways geometric lock-in can fail; the other is fate (c)."
- Table 1: change the caption to "… that are certified to escape"
- Sections 1 and 3: write "explains why amplifications are frequent"
- Section 5: call capped counts lower bounds wherever they appear

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CATEGORY D: TRANSPARENCY AND INTEGRITY

Score: 8/10

Evaluation:

Transparency is a clear strength:

- The scope labels S1–S3 (Section 2) map one to one onto the limitations discussed in Section 6.
- Fate (c) is identified as the main open problem rather than glossed over.
- The census is presented as giving lower bounds, and the undecided seeds are characterized honestly as "candidates for fate (c) or for very long transients".
- AI assistance is disclosed.

The assumptions are explicit almost throughout. The coprimality assumption A1 is justified by [2, Theorem 4], Theorem 3 states its hypothesis $N_0^2 \neq 1$, and every computational cap is listed. The gaps are:

- the implicit assumption that all factors used in certificates are genuinely prime;
- no discussion of the dependence on [2] and [3], which carry correction notices;
- no note that the evidence for Conjecture 9 comes from boxes up to 200 only.

Key Strengths:

- Specific, non-defensive limitations tied to explicit scope labels
- Explicit mathematical hypotheses at every statement

Areas for Improvement:

- Extend S3 in Section 6 with the primality and software assumptions and with the range of the evidence for Conjecture 9
- State which versions of [2] and [3] the paper relies on

CATEGORY E: CONTRIBUTION AND IMPACT

Score: 7/10

Evaluation:

The paper's novelty is clear:

- the first proofs that seeds of this recurrence escape geometric lock-in (Theorem 6), with infinitely many examples (Theorem 7)
- a finite certificate (Theorem 3) for a property that no step limit can decide
- the obstruction 30 | N for escaping orbits and the return of the golden ratio as their growth rate
- a census that separates certified escapes from undecided seeds, and an answer to the transient question of [3, Section 6.5]

The tools are classical (arithmetic in $\mathbb{Z}[\varphi]$ modulo r and Pisano periods), and the setting is the author's own series, so the contribution is substantial within its area but specialized. Its broader reach depends on whether the certificate idea transfers to other recurrences that are linear between congruence-triggered events, which the paper does not yet explore.

Key Strengths:

- A decisive answer to the series' central open question, with a transferable proof technique

Areas for Improvement:

- Add 2–3 sentences in Section 6 on which other recurrences admit the same certificate
- Optionally, note the recurring twin-prime pattern of the certificates (179 and 181; 1,359,179 and 1,359,181; a 22-digit pair for p = 1419)

Summary of Score Breakdown by Category:

Category	Score	Maximum
Conceptual Foundation	22	30
Methodological Rigor	26	30
Evidence & Reasoning	17	20
Transparency and Integrity	8	10
Contribution and Impact	7	10
TOTAL	**80**	**100**

EDITORIAL DECISION

Decision: MINOR REVISIONS

Score Band: 80/100

Decision Thresholds	Range
Accept	90–100
Minor Revisions	80–89 ← This paper
Major Revisions	70–79
Major Revisions (Uncertain)	60–69
Reject (May Resubmit)	50–59
Reject	0–49

Factors Considered in Decision:

- Overall quality score: 80/100
- Citation integrity status: PASS (no impact)
- Plagiarism assessment: PASS (no impact)
- Pattern: very strong rigor and evidence; weaker breadth of literature, and some imprecise statements and reporting
- Feasibility: all required changes are corrections or additions to the text; no new mathematics is needed
- Significance: the first proofs of escape and a finite certificate, in a specialized area

Detailed Justification

This paper scores 80/100, at the threshold of the Minor Revisions band. The mathematics is correct: every proof was checked, and the theorems, Table 1 and both census boxes of Table 2 were reproduced independently. The score reflects how the work is framed and reported, not any defect in its results.

The revisions required are specific:

- correct the sentence in Section 2 that equates escape with the failure of lock-in;
- label the census figures by protocol, correct the caption of Table 1, and call capped counts lower bounds;
- document the primality standard and the software behind the certificates.

None of these requires new mathematics or new computation. The theorems, proofs and conclusions will remain unchanged.

Both integrity checks passed without concerns. All 17 references are cited and genuine, and the adjusted similarity is below 1%. This gives high confidence that the score reflects the paper's scholarly merit.

The contribution justifies publication: a decisive answer to an open question, a finite decision procedure for a property that simulation cannot settle, and an explicit infinite family of escaping seeds. Its reach is currently specialized, and the recommended additions on related literature and transferability would broaden its relevance. We expect the revised paper to meet the acceptance threshold.

Integration of All Evaluation Steps:

- **Step 1 (citation integrity):** a clean, fully verified bibliography, with two small attribution items.
- **Step 2 (plagiarism):** original text; the only overlap is one definitional sentence from the author's own earlier paper.

- **Step 3 (quality evaluation):** very strong rigor and evidence. Improvements are needed in the precision of some statements, the labelling of census figures, the documentation of the computations and the breadth of the literature.

Together, these support a Minor Revisions decision with high confidence of acceptance after revision.

REVISION REQUIREMENTS

Critical Issues (Must Address for Acceptance)

Priority 1 — Essential:

1. **Correct the Statement Equating Escape with Failure of Lock-in**
   - **Current Problem:** Section 2, after Definition 1: "Escape is therefore exactly the failure of geometric lock-in." By Definition 1, seeds of fate (c) also fail to lock in, and the paper keeps fate (c) open (S2, Section 6).
   - **Required Fix:** Rewrite the sentence, for example: "Escape is one of the two ways geometric lock-in can fail; the other is fate (c)."
   - **Success Criterion:** No statement in the paper equates escape with the failure of lock-in.
   - **Importance:** The three-fate framework is the conceptual core of the paper, and this sentence contradicts it.
2. **Make the Census Reporting Protocol-Explicit**
   - **Current Problem:**
     - The Abstract and Conclusions report 956 certified escapes and 2,320 undecided seeds (the extended check in Section 5), while Table 2 reports 907 and 2,369 (1,200 steps, 30-digit cap), without saying so.
     - The protocol behind "for the seeds (1, p) with $p \leq 2{,}000$, 43 escape" (Section 5) is not stated.
     - The caption of Table 1 ("that escape") implies a complete list, although several seeds in its range, for example $p = 45$ (Figure 1), remain undecided.
   - **Required Fix:**
     - Label each reported count with its protocol, for example "907 (Table 2) and 956 after the extended check".
     - Change the caption of Table 1 to "… that are certified to escape".
     - Describe counts obtained under computational caps as lower bounds wherever they appear.
   - **Success Criterion:** Every count can be traced to a stated protocol, and no caption or summary implies completeness.
   - **Importance:** Readers must be able to tell which figures are complete and which are lower bounds.
3. **Document the Computations Behind the Certificates**
   - **Current Problem:** Section 5 does not state the primality standard for the factors used in the census certificates, or the software and versions used (Python; PARI/GP, which is also uncited).
   - **Required Fix:** Add a short "Computational details" paragraph that covers:
     - the software and versions, with a citation for PARI/GP;

    - whether large factors were proved prime or accepted as probable primes (for example PARI/GP's factor_proven setting);
    - how the extended check proves that a seed amplifies again.
  - **Success Criterion:** A reader can judge the rigor of every certificate reported in the census.
  - **Importance:** A certificate is valid only if every factor is genuinely prime; a composite mistaken for a prime could produce a false verdict that a seed is never amplified again.

Important Issues (Strongly Recommended)

Priority 2 — Significantly Improve Paper:

1. **Correct the Remark on the Three Cycles**
   - **Current State:** Section 3: "Applied to the three cycles of [4, Theorem 10], Theorem 3 predicts their amplifying step q − 1." The fixed point (1, 2) has norm −1, outside the hypothesis $N_0^2 \neq 1$.
   - **Recommended Improvement:** Restrict the remark to (3, 5) and (29, 47), and note that (1, 2) is covered by Lemma 2(c).
   - **Expected Benefit:** Removes a hypothesis violation.
2. **Justify the Norm of Fibonacci Pairs Directly**
   - **Current State:** The proof of Proposition 4(a) cites Lemma 1 for the norm of (F_j, F_{j+1}).
   - **Recommended Improvement:** Cite Cassini's identity, which gives $F_{j+1}^2 - F_jF_{j+1} - F_j^2 = (-1)^j$ directly.
   - **Expected Benefit:** A precise justification.
3. **Unify the Step and Index Conventions**
   - **Current State:** Steps are counted from 0, states are indexed from n = 2, and Figure 1 ("escapes after step 10") and Theorem 6(b) (last amplification at step 9) use different counts.
   - **Recommended Improvement:** Fix one convention, state it once in Section 2, and align Figure 1 and Theorem 6(b).
   - **Expected Benefit:** Removes a source of confusion for readers.
4. **Broaden the Literature Context**
   - **Current State:** The literature on orders of φ modulo primes, Fibonacci primitive roots and Artin-type densities is not cited, and no other gcd-driven recurrences are mentioned.
   - **Recommended Improvement:** Cite this literature at Proposition 4 and Conjecture 9, and mention one or two other gcd-driven recurrences in Section 1 (see Recommended Resources).
   - **Expected Benefit:** Places the full primes and Conjecture 9 within existing theory and broadens the readership.
5. **Improve Self-Containedness**
   - **Current State:** Basic facts such as $C_{ka,kb} = kC_{a,b}$, the cofactor factorization and the equivalence of lock-in with periodicity are imported from [2]–[4] without justification, and [2] and [3] carry correction notices.
   - **Recommended Improvement:** Add one-line justifications, and state which versions of [2] and [3] are cited.

- **Expected Benefit:** The paper can be read and checked on its own, independently of later changes to the earlier papers.

Minor Issues (Suggested for Polish)

Priority 3 — Enhancement:

1. References: complete [12] with its editors and full book title; give [7] as 2008/2009; add the AMS identifier for [6] and an https link for [16]; avoid repeating the year as the volume in [2]–[4]; add the GitHub URL to [17].
2. Abstract: reword the first sentence, which closely follows the abstract of [4], so that the two abstracts are distinct.
3. Sections 1 and 3: write "explains why amplifications are frequent" instead of "explains why lock-in is common".
4. Theorem 3: when J_r is nonempty, its difference equals $\pi(r)$; "divides" is correct but weaker than needed.
5. Section 5: consider lifting the 10^{12} cap on discrete logarithms. Pohlig–Hellman in the subgroup generated by φ also certifies (1, 1419) and (1, 1635), raising the (1, p) count for $p \le 2{,}000$ from 43 to 45.
6. Section 6 (S3): note that the evidence for Conjecture 9 comes from boxes up to 200 only.

Revision Timeline and Expectations

Timeline: Less than 1 week (if AI-assisted); 2–4 weeks otherwise
Scope: Small, focused improvements to address specific issues
Re-review: Rapid review of revised sections only
Acceptance Probability: 95%+ if all feedback is addressed
Guidance: The changes are corrections and additions to the text; the paper's theorems and proofs will not need modification.

COMPREHENSIVE FEEDBACK FOR AUTHORS

What This Paper Does Well

This is a compact and genuinely elegant piece of mathematics. The key observation is that between amplifications the dynamics is linear: the shifted state evolves by the Fibonacci matrix and its golden integer is multiplied by φ. From there, Lemma 2 and Theorem 3 turn a question about infinitely many steps into finitely many orbit tests modulo the primes dividing $N_0^2 - 1$. The proof that the seed (11, 5) is never amplified needs only the primes 179 and 181 and can be checked by hand. The construction of Theorem 7 then yields infinitely many escaping seeds. Section 3 shows convincingly why such a certificate is needed: one state first amplifies only at step 47,359.

The paper is also exemplary in its transparency and reproducibility:

- Its scope labels delimit every claim, and its limitations section takes them up one by one.
- The open fate (c) is stated honestly as the main open problem.
- The computational parameters are stated in full, and an implementation written from the text alone reproduced Table 2 exactly.
- The code is archived with a persistent identifier, the references are clean and genuine, and AI assistance is properly disclosed.

Priority Improvements Needed

Highest Priority: Correct the sentence in Section 2 that equates escape with the failure of geometric lock-in, and make the census reporting protocol-explicit. Readers should be able to tell at a glance which counts come from Table 2 and which from the extended check, and that capped counts are lower bounds. The caption of Table 1 should say "certified to escape".

Secondary Priority: Add a short paragraph on computational details: software and versions, the primality standard for factors, and how the extended check proves further amplifications. This is what makes the census certificates fully trustworthy. Please also cite PARI/GP and state which versions of [2] and [3] you rely on.

Additional Considerations: Each of the following would improve the paper further: correcting the remark on the three cycles and the citation in the proof of Proposition 4(a); unifying the index conventions; citing the Artin-type and Fibonacci primitive-root literature at Proposition 4 and Conjecture 9; and a few sentences on which other recurrences admit the same certificate.

Guidance for Revision Process

We recommend addressing the three Priority 1 items first. They are the required changes, and all of them are text-level. The census labels can be taken directly from your existing outputs, and the computational-details paragraph needs only a few sentences.

The Priority 2 and 3 items are recommended rather than required, and most take a sentence or two each. If you use AI assistance, it is well suited to drafting the literature paragraph and to cross-checking every reported count against your script's output. Please verify every citation and number yourself, as you have done throughout this paper.

When resubmitting, please give the location of each change in a point-by-point response. This will allow a rapid re-review of the revised sections only.

Recommended Resources

- Shanks, D. (1972). Fibonacci primitive roots. *The Fibonacci Quarterly*, 10(2), 163–168, 181. The classical starting point for primes at which φ has maximal order, which is exactly the "full prime" condition of Proposition 4.
- Moree, P. (2012). Artin's primitive root conjecture – a survey. *Integers*, 12(6), 1305–1416. https://arxiv.org/abs/math/0412262. Covers density results and heuristics for orders of fixed elements modulo primes, including quadratic units; relevant to Conjecture 9.
- Rowland, E. S. (2008). A natural prime-generating recurrence. *Journal of Integer Sequences*, 11, Article 08.2.8. A well-known gcd-driven recurrence, useful as a comparison point in Section 1.
- Lagarias, J. C., Rains, E. M., & Sloane, N. J. A. (2002). The EKG sequence. *Experimental Mathematics*, 11(3), 437–446. Another gcd-defined sequence whose long-run behaviour is analysed rigorously.
- Pohlig, S., & Hellman, M. (1978). An improved algorithm for computing logarithms over GF(p) and its cryptographic significance. *IEEE Transactions on Information Theory*, 24(1), 106–110. The discrete-logarithm method that lifts the 10^{12} cap in Section 5.
- The PARI Group. *PARI/GP* (cite the version used). Université de Bordeaux. https://pari.math.u-bordeaux.fr/. Includes the factor_proven setting, which controls whether factors are proved prime.

RESUBMISSION PROCESS

When resubmitting your revised manuscript, please provide:

1. Revised Manuscript (DOCX format) with updated version number

- All changes clearly incorporated
- Line numbers enabled
- Following AIR formatting guidelines

2. Cover Letter

- Brief summary of changes made
- Any significant decisions or departures from feedback

3. Point-by-Point Response Document For each point raised in this evaluation:

- List the specific issue/suggestion
- Your response explaining how you addressed it
- Changes made, with specific page/line numbers in the revised manuscript
- If a suggestion is not followed, a clear explanation why

4. Submission Details

- Send as Reply to the same email address
- Subject line: RESUBMISSION - AIR-2026-000983-V2
- Include all three documents listed above as DOCX files

5. Resubmission Fees:

- First resubmission within 7 days: FREE
- After 7 days or 2nd+ resubmissions: $10

CLOSING REMARKS

This paper answers the question its predecessor left open, and answers it with a proof rather than more computation. It gives a finite certificate for a property that no step limit can settle, explicit escaping seeds that can be checked by hand, and computations that we could reproduce exactly from the text alone. These are exactly the qualities a strong mathematical paper should have.

The revisions requested are focused and mostly concern precision of statement and reporting. We are confident they are straightforward to make and will bring the paper to the acceptance threshold. We look forward to receiving the revised manuscript.

NEXT STEPS FOR AUTHORS

Revision Instructions

Immediate Actions:

1. Review all feedback carefully
2. Develop a detailed revision plan
3. Address all critical issues (Priority 1)
4. Strongly consider all Priority 2 issues
5. Improve Priority 3 issues as feasible

Timeline: Less than 1 week (AI-assisted) / 2–4 weeks (unassisted)

We wish you success in your research endeavors.

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